Math 10850, fall 2019, University of Notre Dame

Notes on final exam

December 10, 2019

When and where

The final exam will take place on

Monday, December 16, 4.15pm to 6.15pm

in the usual classroom, Hayes-Healy 231.

What you need to know

Math 10850 is a course where virtually every new topic builds in some fundamental way on previous topics. In that sense, as an initially preparation for the final, you will need to know everything that you knew for the first midterm, and the second midterm, and everything that we have covered since the second midterm.

Practically, the exam will not be cumulative, but rather will focus on everything that we have covered since defining a function. That means there will be no questions on the logic portion at the start of the semester, and no questions on proving properties of the real numbers directly from the axioms (The exception here is the completeness axiom, which we discussed in some detail closer to the end of the semester than the beginning, and so is fair game for the final). It is quite possible, however, that there will be a question that requires using induction to prove some property of functions, limits or derivatives. (Even though induction came up early in the semester, it is fundamental enough that I consider it fair game.)

Here is a summary of the topics that you should prepare for the exam.

Functions Definition, domain, range, codomain, new functions from old, polynomials, rational functions, composition, geometric definition of sin and cos, Dirichlet function, graphs of functions

Limits ε-δ definition, calculating limits directly from definition, showing non-existence of limits, the function $\sin(1/x)$ and variants, sum/product/reciprocal theorem, limits of rational functions, one-sided limits, connection between one-sided limits and limits, infinite limits, one-sided infinite limits, limits at infinity, limit theorems for limits at infinity, limits at infinity of rational functions, the basic trigonometric limit $(\sin x)/x$ near 0

Continuity Definition, continuity of basic functions, continuity of compositions, the Stars over Babylon function, one-sided continuity, continuity on an interval, continuous and positive at a point implies positive on small interval around point, the Intermediate Value Theorem and variants, existence and uniqueness of positive nth roots of positive numbers, continuity of nth root function, odd-degree polynomials have at least one zero, definitions of being bounded and achieving maximum/minimum, continuous functions are locally bounded, Extreme Value Theorem, even-degree polynomials with positive leading coefficient have absolute minimum

Differentiation Definition, motivation via instantaneous velocity, motivation via slopes of tangent lines, computing the derivative directly from the definition, one-sided derivatives and piecewise-defined functions, the derivative as a function, higher derivatives, the linearization, the derivative of power functions, trigonometric functions & root functions, differentiability implies continuity, derivatives of sums, differences & constant multiples, product rule, product rule for product of n functions, reciprocal rule, quotient rule, chain rule & motivation via linearization, composing multiple functions

Applications of the derivative Maximum and minimum points (local and otherwise), Fermat principle, critical points, the possible locations of maximum and minimum points on a closed interval, Mean Value Theorem, Rolle's theorem & motivation, zero derivative on interval implies being constant, definitions of increasing, decreasing and monotone, connection between sign of derivative and monotonicity, first derivative test, second derivative test, convexity and concavity, testing for convexity and concavity via the derivative, curve sketching, L'Hôpital's rule (weak form) & motivation via linearization, proof of weak form of L'Hôpital's rule, general form of L'Hôpital's rule

Completeness Upper bounds, lower bounds, least upper bounds, greatest lower bounds & the completeness axiom, $\mathbb{N}$ is not bounded above, the Archimedean property, $1/n$ can be made arbitrarily small, definition of a subset of the reals being dense, density of rationals, density of irrationals.

Questions will involve:

- definitions of basic concepts
- statements of main theorems
- applications of the key definitions and theorems
- the shorter proofs of propositions associated with the key definitions and theorems. While it is good to be quite familiar with the proofs of the main theorems — IVT, EVT, chain rule — these proofs won't directly appear on the exam. The proofs of Rolle's theorem and the Mean Value Theorem are fair game, since these were derived as corollaries of the EVT.

Practice problems

None of these problems have parts that ask for definitions or statements of key theorems. The actual exam questions usually will.

1. Recall that $f : \mathbb{R} \to \mathbb{R}$ is *even* if $f(-x) = f(x)$, and *odd* if $f(-x) = -f(x)$

   (a) Is it possible for a function f to be both even and odd?

   (b) If f is odd (and differentiable), what can be said about f'?

   (c) Let f be arbitrary (not necessarily even, or odd). Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x) + f(-x)$. Show that g is even.

   (d) Let f be arbitrary (not necessarily even, or odd). Show that there is an even function f_e and an odd function f_o with $f = f_e + f_o$.

   (e) With the same set-up as part (d), show that there is a *unique* choice for the pair (f_e, f_o).

2. (a) Show that the natural numbers $\mathbb{N}$ are not bounded above.

   (b) Show that if $\iota > 0$ and $I > 0$ are any two positive numbers, there is $n \in \mathbb{N}$ with $\iota n \geq I$.

   (c) Show that if $\iota > 0$ is any number, there is a natural number such that $1/n < \iota$ for all $n > N$.

3. (a) Let $f(x) = x/(1 + x^2)$. *Directly from the definition of limit* find $\lim_{x\to a} f(x)$, for any real number a.

   (b) Let g be some function, defined near a. Suppose that $\lim_{x\to a} g(x) = L$. Let h be obtained from g, by changing the value of g at finitely many places. *Directly from the definition of limit* show that $\lim_{x\to a} h(x) = L$.

4. (a) Suppose that f is increasing on the interval (a, b), and that $\lim_{x\to b^-} f(x)$ exists. Prove that for any $c \in (a, b)$,

   $$\lim_{x\to b^-} f(x) > f(c)^1.$$

   (b) Prove that if f is increasing on (a, b) and continuous at both a and b, then f is increasing on $[a, b]$.

5. Recall the factorization formula $X^3 - Y^3 = (X - Y)(X^2 + XY + Y^2)$.

   (a) Use the intermediate value theorem to show that for every real number a, there is a unique real number $\sqrt[3]{a}$ with $(\sqrt[3]{a})^3 = a$.

   (b) Show that the function $c : \mathbb{R} \to \mathbb{R}$ defined by $c(x) = \sqrt[3]{x}$ is continuous everywhere.

   (c) Find where c is differentiable, and (from the definition of derivative) find c'.

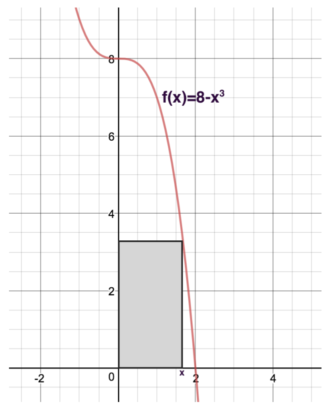
[1] We used this fact in our sketch of the proof that convexity implies increasing derivative

(d) Show, using the derivative, that c is an increasing function.

6. (a) Let f be a function that is continuous on $[a, \infty)$ and differentiable on (a, ∞). Prove that if $f'(x) \geq 0$ for all $x > a$, then $f(x) \geq f(a)$ for all $x \geq a$.

   (b) *Use the previous part* to prove the following inequality: if n is a positive integer, then for all $x \geq 0$
$$(1+x)^n \leq 1 + nx(1+x)^{n-1}.$$

7. A rectangle is drawn in the first quadrant (where both x and y co-ordinates are non-negative), with one corner at $(0, 0)$, the diagonally opposite corner on the graph of the function $f(x) = 8 - x^3$, and sides parallel to the axes.



   (a) Express the area $A(x)$ of the rectangle as a function of x, the length of the base of the rectangle.

   (b) What is the domain of A? That is, what is the range of values for x that lead to rectangles that satisfy the conditions of the question? (Include the rectangle with base 0, and the rectangle with height 0).

   (c) What is the maximum possible area? *Carefully* justify your conclusion.

8. Without actually calculating $\sqrt{66}$, prove that
$$8\,\frac{1}{9} \leq \sqrt{66} \leq 8\,\frac{1}{8}.$$

9. Suppose that f is defined on $(0, \infty)$, and that f' is increasing. Let $g(x) = f(x)/x$.

   (a) By considering the function $f(x) = 1 + x^2$, show that it is not necessarily the case that g is increasing on $(0, \infty)$.

   (b) Prove that if $f(0) = 0$, then g is increasing on $(0, \infty)$. (**Hint**: try to show that g' is positive. The MVT may be helpful in this.)

10. (a) Prove that for $0 \leq a \leq b$, it is always the case that
$$a \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b.$$

(b) Suppose that N is a perfect power of 2, say $N = 2^n$ with $n \geq 1$. Prove by induction on n that if $a_1, a_2, \ldots, a_N$ are any N non-negative numbers then

$$(a_1 a_2 \cdots a_N)^{\frac{1}{N}} \leq \frac{a_1 + a_2 + \cdots + a_N}{N}.$$

11. A number a is said to be a *root* of a polynomial p if there is another polynomial q such that $p(x) = (x - a)q(x)$ for all x. A number a is said to be a *double root* of a polynomial p if there is another polynomial q such that $p = (x - a)^2 q(x)$ for all x.

   (a) Suppose that a is a double root of the polynomial p. Prove that a is both a root of p and a root of p' (the derivative of p).

   (b) Suppose that a is both a root of the polynomial p and a root of the polynomial p'. Prove that a is a double root of p.

12. Suppose that functions f and g are related by the following differential equation:

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for all real x. Prove that if f is equal to zero at two distinct numbers a, b, then f is equal to 0 at all numbers between a and b.

13. (a) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that f must have a fixed point. That is, prove that there exists at least one $x \in [0, 1]$ such that $f(x) = x$.

   (b) Let $f : [0, 1] \to [0, 1]$ be a continuous function that is differentiable on $(0, 1)$, with $f'(x)$ not equal to 1 for any $x \in (0, 1)$. Show that the fixed point of f (found in part (a)) is unique.

14. In this question, the number a is some real number *strictly* between 0 and 4 ($0 < a < 4$).

   (a) Show (using derivatives) that for all real x, $x^2 - ax + a > 0$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{x^2 - ax + a}.$$

   (c) Show that there is a number A (which depends on a) such that f is increasing on $(-\infty, A)$ and decreasing on (A, ∞).

   (d) By the first derivative rest, f has maximum value $f(A)$. What choice of $a \in (0, 4)$ makes $f(A)$ as *small* as possible?