Math 10850, Honors Calculus 1

Homework 9

Solutions

1. When we motivated the definition of derivative via instantaneous velocity, we got to the expression $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, and when we motivated it by slope of tangent line, we ended up with $\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$. It seems intuitively clear that these two expressions are the same. This question asks you to prove this, directly from the definition of limit. It also asks you to show that the expression we got at the end of the proof that differentiability implies continuity, namely $\lim_{h\to 0} f(a+h) = f(a)$, does indeed imply that f is continuous at a, even though the expression looks a little bit different from the definition of continuity.

   (a) Let g be a function defined near a. Suppose that $\lim_{b\to a} g(b)$ exists and equals L. Prove that $\lim_{h\to 0} g(a+h)$ exists and also equal L.[1]

   Solution: Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that

   $$0 < |b-a| < \delta \text{ implies } |g(b) - L| < \varepsilon. \ (\star)$$

   Now suppose $0 < |h| < \delta$ (for the *same* δ). This says that $0 < |(a+h)-a| < \delta$, so, applying $(\star)$ (with $b = a+h$) we get that $|g(a+h) - L| < \varepsilon$, which is what was required to prove that $\lim_{h\to 0} g(a+h) = L$.

   (b) **OPTIONAL!** (Almost exactly the same as part (a)) Let g be a function defined near a. Suppose that $\lim_{h\to 0} g(a+h)$ exists and equals L. Prove that $\lim_{b\to a} g(b)$ exists and also equal L.

   Solution: Let $\varepsilon > 0$ be given. There is $\delta > 0$ such that

   $$0 < |h| < \delta \text{ implies } |g(a+h) - L| < \varepsilon. \ (\star\star)$$

   Now suppose $0 < |b-a| < \delta$ (for the *same* δ). Applying $(\star\star)$ (with $h = b-a$) we get that $|g(a+(b-a)) - L| < \varepsilon$, or, equivalently, $|g(b) - L| < \varepsilon$, which is what was required to prove that $\lim_{b\to a} g(b) = L$.

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[1] You <u>must</u> use the ε-δ definition of the limit here. Start by supposing that an $\varepsilon > 0$ is given. You

- **Know**: that for any $\varepsilon' > 0$ there is $\delta' > 0$ such that $0 < |b-a| < \delta'$ implies $|g(b) - L| < \varepsilon'$.

You

- **Want**: that there is $\delta > 0$ such that $0 < |h| < \delta$ implies $|g(a+h) - L| < \varepsilon$.

Explain in your proof how to get where you **want** from what you **know**.

(c) Parts (a) and (b) together show that if g is a function defined near a, then if one of $\lim_{h\to 0} g(a+h)$, $\lim_{b\to a} g(b)$ exists, they both exist, and are equal. Apply this to show that our two definitions of the derivative are equivalent. That is, show that if f is a function defined at and near a, and if either one of $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, $\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$ exists, then they both exist and are equal. (This should be a simple matter of finding the right choice of function g).

Solution: For each, define the function g by

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

Note that

$$g(a+h) = \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad g(b) = \frac{f(b) - f(a)}{b - a}$$

We have that

- $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists and equals L

if and only if

- $\lim_{h\to 0} g(a+h)$ exists and equals L,

if and only if

- $\lim_{b\to a} g(b)$ exists and equals L,

if and only if

- $\lim_{b\to a} \frac{f(b)-f(a)}{b-a}$ exists and equals L.

(d) **OPTIONAL!** (Very similar to part (c)). For f defined at and near a, we defined "f continuous at a" to mean "$\lim_{x\to a} f(x) = f(a)$". Show that an equivalent definition is "$\lim_{h\to 0} f(a+h) = f(a)$".

Solution: This follows immediately from parts (a) and (b): just take $g = f$ and $L = f(a)$.

2. Let f be defined by $f(x) = \frac{x+1}{x-1}$. *Directly from the definition*[2] calculate $f'(a)$ for each $a \neq 1$.

[2]The definition involves a limit; you can assume any facts/theorems we have proven about limits and continuity. This note also applies to the next question.

Solution: For each $a \neq 1$ and $h \neq 0$ we have

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\frac{a+h+1}{a+h-1}-\frac{a+1}{a-1}}{h} \\
&= \frac{(a+h+1)(a-1)-(a+1)(a+h-1)}{h(a+h-1)(a-1)} \\
&= \frac{a^2-a+ah-h+a-1-(a^2+ah-a+a+h-1)}{h(a+h-1)(a-1)} \\
&= \frac{a^2+ah-h-1-(a^2+ah+h-1)}{h(a+h-1)(a-1)} \\
&= \frac{a^2+ah-h-1-a^2-ah-h+1}{h(a+h-1)(a-1)} \\
&= \frac{-2h}{h(a+h-1)(a-1)} \\
&= \frac{-2}{(a+h-1)(a-1)}.
\end{aligned}$$

Since

$$\lim_{h\to 0} \frac{-2}{(a+h-1)(x-1)} = \frac{-2}{(a-1)^2}$$

we get that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \frac{-2}{(a-1)^2},$$

so f is differentiable for all $a \neq 1$, with derivative

$$f'(a) = \frac{-2}{(a-1)^2}.$$

3. (a) Prove, *directly from the definition* that if $f_3(x) = x^{1/3}$ then for all $a \neq 0$,

$$f_3'(a) = \frac{1}{3a^{2/3}}.$$

(Here $x^{2/3}$ is defined to be $(x^{1/3})^2$. You may find the factorization of $X^3 - Y^3$ helpful.)

Solution: Fix $a \neq 0$. We have $\frac{f_3(a+h)-f_3(a)}{h}$

$$\begin{aligned}
&= \frac{(a+h)^{1/3} - a^{1/3}}{h} \\
&= \left(\frac{(a+h)^{1/3} - a^{1/3}}{h}\right)\left(\frac{(a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}}{(a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}}\right) \\
&= \frac{\left((a+h)^{1/3} - a^{1/3}\right)\left((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}\right)}{h\left((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}\right)} \\
&= \frac{\left((a+h)^{1/3}\right)^3 - \left(a^{1/3}\right)^3}{h\left((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}\right)} \quad \text{(using } X^3 - Y^3 = (X-Y)(X^2+XY+Y^2)\text{)} \\
&= \frac{(a+h) - a}{h\left((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}\right)} \\
&= \frac{h}{h\left((a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}\right)} \\
&= \frac{1}{(a+h)^{2/3} + (a+h)^{1/3}a^{1/3} + a^{2/3}}.
\end{aligned}$$

By continuity of the cube root function (and the sum-product-reciprocal theorem for limits), this last expression tends to

$$\frac{1}{3a^{2/3}}$$

as h tends to 0. This proves that

$$f_3'(a) = \frac{1}{3a^{2/3}}.$$

(b) Is f_3 differentiable at 0?

Solution: We have

$$\begin{aligned}
\frac{f_3(0+h) - f_3(0)}{h} &= \frac{h^{1/3} - 0}{h} \\
&= \frac{1}{h^{2/3}}.
\end{aligned}$$

This does not tend to a limit as h approaches 0 (it grows arbitrarily large and positive as $h \to 0^+$ and arbitrarily large and negative as $h \to 0^-$), so $f_3'(0)$ does not exist.

(c) **OPTIONAL!** Let $n \geq 2$ be a natural number, and let f_n be defined by $f_n(x) = x^{1/n}$ (so the domain of f_n is all reals if n is odd, and all non-negative reals if n is even). Prove, directly from the definition, that

$$f_n'(a) = \frac{1}{na^{(n-1)/n}}$$

for all $a \neq 0$ (in the domain of f_n), and that f_n is not differentiable at 0 for any n.

Solution: Using the factorization

$$X^n - Y^n = (X - Y)(X^{n-1} + X^{n-2}Y + \cdots + XY^{n-2} + Y^{n-1}),$$

and following the same argument as in part (a), we find that for any $a \neq 0$ (and $a > 0$ also, in the case n even)

$$\frac{f_n(a+h) - f_n(a)}{h} = \frac{1}{(a+n)^{\frac{n-1}{n}} + (a+n)^{\frac{n-2}{n}} a^{\frac{1}{n}} + \cdots + (a+n)^{\frac{1}{n}} a^{\frac{n-2}{n}} + a^{\frac{n-1}{n}}}.$$

This latter approaches

$$\frac{1}{na^{\frac{n-1}{n}}}$$

as h approaches 0, so

$$f_n'(a) = \frac{1}{na^{(n-1)/n}}$$

for all $a \neq 0$ (in the domain of f_n).

On the other hand, for odd n

$$\begin{aligned} \frac{f_n(0+h) - f_n(0)}{h} &= \frac{h^{1/n} - 0}{h} \\ &= \frac{1}{h^{\frac{n-1}{n}}}. \end{aligned}$$

This does not tend to a limit as h approaches 0 (it grows arbitrarily large and positive as $h \to 0^+$ and arbitrarily large and negative as $h \to 0^-$), so $f_n'(0)$ does not exist.

For even n, we need to consider a one-sided limit: for $h > 0$

$$\begin{aligned} \frac{f_n(0+h) - f_n(0)}{h} &= \frac{h^{1/n} - 0}{h} \\ &= \frac{1}{h^{\frac{n-1}{n}}}. \end{aligned}$$

This does not tend to a limit as h approaches 0 from above (it grows arbitrarily large and positive), so for even n, $(f_n)'_+(0)$ does not exist.

4. Find f' if $f(x) = [x]$ (remember that $[x]$ is the *integer part* of x: the greatest integer less than or equal to x).

   Solution: Let $f(x) = [x]$ (domain: all reals). Let a be given. Suppose first that a is not an integer. Then in some small interval around a, $f(a)$ is constant, always taking the value $[a]$, and so

   $$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[a] - [a]}{h} = \lim_{h \to 0} 0 = 0,$$

   so $f'(a) = 0$.

Now suppose a is an integer. Then for (small) $h > 0$, $f(a+h) = a = f(a)$ and for (small) $h < 0$, $f(a) = a - 1 \neq f(a)$. So

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0^+} \frac{a-a}{h} = \lim_{h\to 0^+} 0 = 0,$$

while

$$\lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0^-} \frac{(a-1)-a}{h} = \lim_{h\to 0^-} \frac{-1}{h},$$

which does not exist. It follows that f is not differentiable at these a. So, in summary, f' is the function whose domain is all non-integer reals, that is constantly 0 on its domain.

5. Imagine a road on which the speed limit is specified at every single point. That is, there is a certain function L such that the speed limit x miles from the beginning of the road is $L(x)$.

   Two cars, A and B, are traveling along the road. A's position at time t is $a(t)$, and B's is $b(t)$.

   (a) What equation expresses the fact that A always travels at the speed limit? (Be careful — question your first answer!)

   Solution: The answer is *not* $a'(t) = L(t)$! The equation is $a'(t) = L(a(t))$ (A's speed at time t is the same as the speed limit at A's *position* at time t, not the same as the speed limit at time t).

   (b) Suppose A always goes at the speed limit, and B's position at time t is always A's position at time $t-1$. Show that B is also going at the speed limit at all times.

   Solution: Because A is always going at the speed limit, we have $a'(t) = L(a(t))$. We also have $b(t) = a(t-1)$. Now this says that

$$b'(t) = a'(t-1)^3 = L(a(t-1)) = L(b(t)),$$

   which is exactly the equation that encodes that B is going at the speed limit at all times.

   (c) Suppose, instead, that B always stays *a constant distance c behind A*. Under what conditions will B always be traveling at the speed limit?

[3] Why?

$$\begin{aligned} b'(t) &= \lim_{h\to 0} \frac{b(t+h) - b(t)}{h} \\ &= \lim_{h\to 0} \frac{a(t+h-1) - a(t-1)}{h} \\ &= \lim_{h\to 0} \frac{a([t-1]+h) - a([t-1])}{h} \\ &= a'(t-1). \end{aligned}$$

Solution: Again, we have that $a'(t) = L(a(t))$. But now the relation between functions a and b is that there is some constant $c > 0$ such that $b(t) + c = a(t)$, which says that

$$b'(t) = a'(t) = L(a(t)) = L(b(t) + c).$$

Since the condition for B to always be going at the speed limit is $b'(t) = L(b(t))$, we must have that $L(b(t) + c) = L(b(t))$ at all times. This says that the speed limit is a periodic function, with period c (more correctly: with period c' where c/c' is an integer).

Here's a more intuitive way to think about it: Car A is being followed by Car B, who is always c meters behind Car A. Car A is always going at the speed limit (at Car A's current location) and because Car B is a constant distance behind, Car B is at all times going at the same speed as Car A. So the condition under which Car B is always going at the speed limit is: the speed limit is the same at any two points distance c meters apart. This condition is satisfied by the constant speed limit, but also by any periodic speed limit whose period divides perfectly into c.

6. (a) Give an example of a function which is continuous at all reals, can be differentiated at all reals, whose derivative is continuous at all reals, but which cannot be differentiated twice at 0. (**Hint**: The function f defined by $f(x) = |x|$ is not differentiable at 0).

Solution: Following the hint, if we could find a function which is continuous at all reals, can be differentiated at all reals, and has derivative $|x|$, then we would be done. Here is one such function:

$$f(x) = \begin{cases} \frac{x^2}{2} & \text{if } x \geq 0 \\ -\frac{x^2}{2} & \text{if } x \leq 0. \end{cases}$$

For $x > 0$ we clearly have $f'(x) = x = |x|$, and for $x < 0$ we have $f'(x) = -x = |x|$. We also easily get $f'_+(0) = f'_-(0) = 0$, so indeed $f'(x) = |x|$ for all $x \in \mathbb{R}$.

(b) For each $k \geq 1$, give an example of a function which is continuous at all reals, can be differentiated k times at all reals, and whose kth derivative is continuous at all reals, but which cannot be differentiated $k + 1$ times at 0.

Solution: There are many possibilities. Here's one. Consider the function defined by

$$f(x) = \begin{cases} x^{k+1} & \text{if } x \geq 0 \\ -x^{k+1} & \text{if } x \leq 0 \end{cases}$$

(notice that when $k = 0$ this is another way of writing $f(x) = |x|$; notice also that there is no ambiguity at $x = 0$ since both clauses of the definition give the same value).

Clearly f is continuous at all reals, and is differentiable infinitely often (not just k times) at all reals other than perhaps at $x = 0$. At $x = 0$, the function is easily seen to be k times. Indeed, using limit results that we have already calculated we can determine[4] that for $m \le k$

$$f^{(m)}(x) = \begin{cases} (k+1)k(k-1)\cdots(k-(m-2))x^{k-(m-1)} & \text{if } x \ge 0 \\ -(k+1)k(k-1)\cdots(k-(m-2))x^{k-(m-1)} & \text{if } x \le 0 \end{cases}$$

In particular

$$f^{(k)}(x) = \begin{cases} (k+1)!x & \text{if } x \ge 0 \\ -(k+1)!x & \text{if } x \le 0, \end{cases}$$

so in fact $f^{(k)}(x) = (k+1)!|x|$. As we have observed in the first part of this question, this function is not differentiable at 0, so f cannot be differentiated $k+1$ times at 0.

7. Recall that $f^{(k)}$ denotes the kth derivative of the function f, and that by convention $f^{(0)}$ means f itself.

   We have

$$(fg)^{(0)} = fg = f^{(0)}g^{(0)},$$
$$(fg)^{(1)} = (fg)' = fg' + f'g = f^{(0)}g^{(1)} + f^{(1)}g^{(0)},$$

   and

$$(fg)^{(2)} = (fg)'' = (fg' + f'g)' = fg'' + 2f'g' + f''g'' = f^{(0)}g^{(2)} + 2f^{(1)}g^{(1)} + f^{(2)}g^{(0)}.$$

   There seems to be a pattern here:

$$(fg)^{(n)} = \sum_{k=0}^{n} (\text{SOME COEFFICIENT DEPENDING ON } n \text{ and } k)\ f^{(k)}g^{(n-k)}.$$

   Find the specific pattern, and prove that is correct for all $n \ge 0$.

   Solution: The pattern is

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k}\ f^{(k)}g^{(n-k)}$$

   where $\binom{n}{k}$ is the binomial coefficient, the same coefficient that appears in the expansion of $(x+y)^n$.

   We prove this by induction on n, with the base case $n = 0$ following from $\binom{0}{0} = 1$.

   For the induction step, assume that for some $n \ge 0$ we have

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k}\ f^{(k)}g^{(n-k)}$$

[4] Probably to be completely precise, this needs to be done by induction, but it's ok not to give that level of precision.

and consider

$$\begin{aligned}
(fg)^{(n+1)} &= (fg^{(n)})' \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)' \qquad \text{(inductive hypothesis)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)} g^{(n-k)}\right)' \qquad \text{(linearity of derivative)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k)} g^{(n-k+1)} + f^{(k+1)} g^{(n-k)}\right) \qquad \text{(product rule)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} + \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n-k+1)} \qquad \text{(shift of index)} \\
&= f^{(0)} g^{(n-k+1)} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) f^{(k)} g^{(n-k+1)} + f^{(n+1)} g^{(0)} \qquad \left(\binom{n}{0} = \binom{n}{n} = 1\right) \\
&= f^{(0)} g^{(n-k+1)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)} g^{(n-k+1)} + f^{(n+1)} g^{(0)} \qquad \text{(Pascal's identity)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} \qquad \left(\binom{n+1}{0} = 1, \quad \binom{n+1}{n+1} = 1\right).
\end{aligned}$$

This completes a proof by induction of the identity.

8. This question will very likely require using the ε-δ definition of the limit.

   (a) Define f as follows:

   $$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   Prove that f is differentiable at 0.

   Solution: If h is rational then

   $$\frac{f(0+h) - f(0)}{h} = \frac{h^2}{h} = h$$

   and if h is irrational then

   $$\frac{f(0+h) - f(0)}{h} = \frac{0}{h} = 0.$$

   We claim that

   $$\frac{f(0+h) - f(0)}{h} \to 0 \quad \text{as} \quad h \to 0$$

(which would show that $f'(0) = 0$). Indeed, given $\varepsilon > 0$, if $0 < |h| < \varepsilon$ then $(f(0+h) - f(0))/h$ either takes value h or 0, so $|(f(0+h) - f(0))/h| \leq |h| < \varepsilon$. This shows, directly from the definition of limit, that $\lim_{h\to 0}(f(0+h) - f(0))/h = 0$ (we are taking $\delta = \varepsilon$ in the definition).

(b) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

Solution: If $|f(x)| \leq x^2$ for all x then in particular $f(0) = 0$ and so

$$\left|\frac{f(0+h) - f(0)}{h}\right| = \frac{|f(h)|}{|h|} \leq \frac{h^2}{|h|} = |h|.$$

Reproducing a part of the proof from above we claim that

$$\frac{f(0+h) - f(0)}{h} \to 0 \quad \text{as} \quad h \to 0$$

(which would show that $f'(0) = 0$). Indeed, given $\varepsilon > 0$, if $0 < |h| < \varepsilon$ then $|(f(0+h) - f(0))/h| \leq |h| < \varepsilon$. This shows, directly from the definition of limit, that $\lim_{h\to 0}(f(0+h) - f(0))/h = 0$ (we are taking $\delta = \varepsilon$ in the definition).

(c) **OPTIONAL!** Generalize part b): find a condition on a function g, such that you can prove the following statement:

> "Let f be a function such that $|f(x)| \leq |g(x)|$ for all x. Then f is differentiable at 0."

Your condition should be satisfied by the function $g(x) = x^2$.

Solution: The most general condition that can be imposed is $g(0) = g'(0) = 0$ (which is certainly satisfied by $g(x) = x^2$). Under these conditions, we claim that

$$\frac{f(0+h) - f(0)}{h} \to 0 \quad \text{as} \quad h \to 0$$

and so $f'(0) = 0$. Indeed, if $|f(x)| \leq |g(x)|$ for all x and $g(0) = 0$ then in particular $f(0) = 0$ and so

$$\left|\frac{f(0+h) - f(0)}{h}\right| = \frac{|f(h)|}{|h|} \leq \frac{|g(h)|}{|h|}.$$

Now since g is differentiable at 0 with derivative 0 it follows that

$$\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{g(h)}{h} = 0,$$

and so $|g(h)|/|h|$ can be made arbitrarily small by choosing h small enough.

9. Prove that if f is an even function (one satisfying $f(x) = f(-x)$ for all x) then f' is odd (satisfies $f'(x) = -f'(-x)$ for all x).

Solution: The quickest way to do this is via the chain rule. Let $g(x) = f(-x) = (f \circ m)(x)$ where m is the function that sends x to $-x$. By the chain rule $g'(x) = f'(m(x))m'(x) = -f'(-x)$. Now since f is even, we have also $g(x) = f(x)$, so $g'(x) = f'(x)$. It follows that $f'(x) = -f'(-x)$, and so f' is odd.

Since we don't yet know the chain rule, we need a more direct proof. We have that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

and

$$-f'(-x) = -\lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = -\lim_{h\to 0} \frac{f(x-h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$$

(using in the second equality that f is even).

So we need to show

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}.$$

Suppose $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = L$. Then given $\varepsilon > 0$ there is $\delta > 0$ such that $0 < |h| < \delta$ implies

$$\left| \frac{f(x+h) - f(x)}{h} - L \right| < \varepsilon.$$

Now if h satisfies $0 < |h| < \delta$, then $h \in (-\delta, \delta)$, and so also is $-h$, so, using

$$\left| \frac{f(x+(-h)) - f(x)}{(-h)} - L \right| = \left| \frac{f(x-h)) - f(x)}{-h} - L \right| = \left| \frac{f(x) - f(x-h)}{h} - L \right|$$

and

$$\left| \frac{f(x+(-h)) - f(x)}{(-h)} - L \right| < \varepsilon$$

we know that

$$\left| \frac{f(x) - f(x-h)}{h} - L \right| < \varepsilon.$$

This shows that $\lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = L$ also.

A similar argument shows that if $\lim_{h\to 0} \frac{f(x)-f(x-h)}{h} = L$ then also $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = L$. So the two limits are equal, as required.

10. (a) If $f + g$ is differentiable at a, are f and g necessarily differentiable at a?

    Solution: Not necessarily. Consider $f(x) = |x|$ and $g(x) = -|x|$, neither of which are differentiable at 0; but the sum $f + g$ is identically 0, so is differentiable at 0.

    (b) If both fg and f are differentiable at a, what conditions on f imply that g is differentiable at a?

    Solution: If $f(a) \neq 0$ then the quotient fg/f is differentiable at a (by the quotient rule), so g is differentiable at a.

    If $f(a) = 0$ then it is not certain that g is differentiable at a. Consider, for example, $g(x) = |x|$, $a = 0$ and $f(x) = 0$.

11. **OPTIONAL!** There are many ways to write the identity function I (defined by $I(x) = x$) as a product fg of two differentiable functions — for example, $f(x) = 3x$ and $g(x) = 1/3$. Is it possible to write $I = fg$ where f and g are both differentiable, and satisfy $f(0) = g(0) = 0$?

    Solution: We claim that it is not possible. For suppose it were and that $I = fg$ is the representation. Differentiating both sides, we obtain the identity $1 = fg' + f'g$. Evaluating both sides at 0, we get $1 = f(0)g'(0) + f'(0)g(0)$. But by hypothesis $f(0) = g(0) = 0$ and so $f(0)g'(0) + f'(0)g(0) = 0$, a contradiction.