Math 10850, Honors Calculus 1

Homework 8

Solutions

1. (Note that this question is *not* about applying the Extreme Value Theorem; the given functions may or may not be continuous, and may or may not be defined on closed intervals.)

   For each of the following functions

   (a) say whether they are bounded above, and/or below on the given interval, and

   (b) whether they achieve their maximum and/or minimum value on the given interval.

   i. $f(x) = x^2$ on $(-1, 1)$.

   Solution:
   - Bounded above (e.g. by 1),
   - bounded below (e.g. by 0),
   - does *not* take on maximum value (can make x^2 arbitrarily close to 1 on $(-1, 1)$, but not equal 1),
   - does take on minimum value 0 at $x = 0$.

   ii. $f(x) = x^2$ on $[0, \infty)$

   Solution:
   - *Not* bounded above (can make x^2 arbitrarily large by taking x arbitrarily large),
   - bounded below (e.g. by 0),
   - does *not* take on maximum value (has no maximum value),
   - does take on minimum value 0 at $x = 0$.

   iii. $f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1/q & \text{if } x = p/q \text{ in lowest terms, } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ on $[0, 1]$

   Solution:
   - Bounded above (e.g. by 1),
   - bounded below (e.g. by 0),
   - does take on maximum value (e.g. at $x = 1$),
   - does take on minimum value 0 (e.g. at $x = \sqrt{2} - 1$).

iv. $f(x) = \begin{cases} x & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$ on $[0, a]$. Here $a > 0$. The answer may depend on a, so you may need to treat cases.

Solution: First suppose that a is rational. Then the function

- is bounded above (e.g. by a),
- is bounded below (e.g. by 0),
- does take on maximum value (at $x = a$),
- does take on minimum value 0 (e.g. at 0).

Next, suppose that a is irrational. Then the function

- is bounded above (e.g. by a),
- is bounded below (e.g. by 0),
- does *not* take on maximum value (can make f arbitrarily close to a on $[0, a]$ by taking rational inputs arbitrarily close to a, but not equal a),
- does take on minimum value 0 (e.g. at 0).

2. For each the following sets

(a) find the least upper bound, and the greatest lower bound, if they exist. Note that the l.u.b. and the g.l.b. are *numbers*, so (at least for the purposes of this question) it is not legitimate to say, for example "$\sup A = \infty$".

(b) Also, in the cases where the l.u.b. and/or g.l.b. exists, say whether these values happen to belong to the sets in question.

i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Solution: L.u.b. is 1, and it is in the set. G.l.b. is 0, and it is not in the set.

ii. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$

Solution: L.u.b. is 1, and it is in the set. G.l.b. is 0, and it is in the set.

iii. $\{x : x^2 + x + 1 \geq 0\}$

Solution: A long time ago we proved that $x^2 + x + 1$ is *always* strictly positive (one possible proof: it's positive when $x = 1$. For $x \neq 1$ it is the same as $(x^3 - 1)/(x - 1)$. For $x > 1$ this is positive divided by a positive, so positive, and for $x < 1$ it is a negative divided by a negative, so positive). So the set in question is $\mathbb{R}$, which is neither bounded above nor below.

iv. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

Solution: This set includes the number 0, the negative numbers

$$-2/3, -4/5, -6/7, -8/9, \ldots$$

and the positive numbers

$$3/2, 5/4, 7/6, 9/8, \ldots.$$

L.u.b. is $3/2$, and it is in the set. G.l.b. is -1, and it is *not* in the set.

3. **OPTIONAL!** (A little bit of history — this was Archimedes' approach to estimating π)

(a) Suppose that $a_1, a_2, \ldots$ is a sequence of positive numbers with $a_{n+1} \leq a_n/2$. Prove that for any $\varepsilon > 0$ there is some n with $a_n < \varepsilon$. (Here I don't want you to make an assertion like "$1/2^n$ can be made arbitrarily small, by making n sufficiently large", without a clear proof. You may assume the fact that we proved in class, that $\mathbb{N}$ is unbounded.)

Solution: By induction it is easy to prove that $a_n \leq a_1/2^{n-1}$ for all n. So it's enough to show that for all positive numbers a_1 and for all $\varepsilon > 0$ there is an n such that $a_1/2^{n-1} < \varepsilon$.

Suppose this were not the case for some $a_1, \varepsilon > 0$. Then for all $n \in \mathbb{N}$ we would have $a_1/2^{n-1} \geq \varepsilon$ or equivalently $2^{n-1} \leq a_1/\varepsilon$.

Now we can prove by induction that for all $n \geq 1$, we have $n \leq 2^{n-1}$ (base case is easy; for induction step, by induction $2^{(n+1)-1} = 2 \cdot 2^{n-1} \geq 2n$, so it is enough to show that $2n \geq n+1$, which is certainly true for $n \geq 1$).

So from $2^{n-1} \leq a_1/\varepsilon$ for all $n \in \mathbb{N}$ we conclude $n \leq a_1/\varepsilon$ for all $n \in \mathbb{N}$, which contradicts the fact that $\mathbb{N}$ is unbounded.

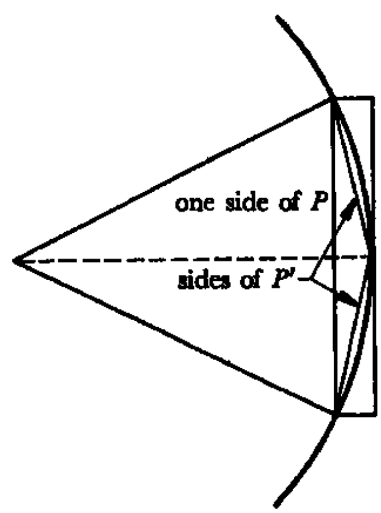
(b) Suppose P is a regular polygon, inscribed inside a circle. If P' is the inscribed regular polygon with twice as many sides as P, show that the quantity

$$\text{area of circle} - \text{area of } P'$$

is less than half the quantity

$$\text{area of circle} - \text{area of } P$$

(see figure below, taken from Spivak Chapter 8).



Solution: Let's say P is an n-gon.

Referring to the figure below marked Figure 7, augmented from Spivak: The difference between the area of P and the area of the circle is n times the area of the circle cap

EAC (the region bounded by the line segment EA and by the portion of the circle from E to A that includes C); call this region A_1, and let its area be a_1, so $\varepsilon_n = na_1$. The difference between the area of P' and the area of the circle is $2n$ times the area of the circle cap that is bounded by the line segment EC and by the portion of the circle from E to C that lies inside $\triangle ECD$; call this region A_2, and let its area be a_2, so $\varepsilon_{n+1} = 2na_2$.

We want to show that $\varepsilon_{n+1} \leq \varepsilon_n/2$, or equivalently that $2na_2 \leq na_1/2$, or equivalently that $4a_2 \leq a_1$. For this it is enough to show that four disjoint regions can be found inside A_1, each of which has area a_2. There is such a set of four regions:

- first, the region A_2, which has area a_2 by definition
- second, the circle cap that is bounded by the line segment AC and by the portion of the circle from A to C that lies inside $\triangle ACB$; by symmetry this has area a_2
- third, the triangle $\triangle ECF$ is congruent to the triangle $\triangle ECD$, so a copy of A_2 can be fitted inside $\triangle ECF$
- and fourth, the triangle $\triangle CAF$ is congruent to the triangle $\triangle CAB$, so a copy of A_2 can be fitted inside $\triangle CAF$.

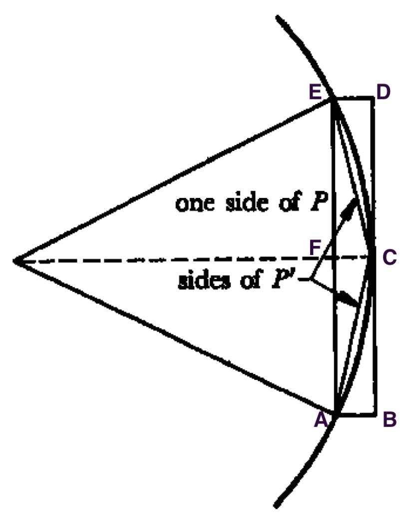


FIGURE 7

(c) Show that for every $\varepsilon > 0$, it is possible to inscribe a regular polygon P into a circle, such that the quantity

$$\text{area of circle} - \text{area of } P$$

is less than ε.[1]

Solution: Start by inscribing a largest possible square P_1 into the circle, and let a_1 be the difference between the area of the circle and the area of the square. Form P_2

[1] Archimedes used this, called the "method of exhaustion", together with an analogous result for *superscribed* polygons, to show $223/71 < \pi < 22/7$.

from P_1 by the process described in part b), and let a_2 be the difference between the area of the circle and the area of P_2. By the result of part b), $a_2 \le a_1/2$. In general, form P_{n+1} from P_n by the process described in part b), and let a_{n+1} be the difference between the area of the circle and the area of P_n. By the result of part b), $a_{n+1} \le a_n/2$.

Now given $\varepsilon > 0$, by the result of part a) there is an n large enough so that $a_n < \varepsilon$. P_n is the required polygon.

4. Suppose that A and B are two non-empty sets of numbers such that $x \le y$ for all $x \in A$ and all $y \in B$.

   (a) Prove that $\sup A \le y$ for all $y \in B$.

   Solution: Given $y \in B$, by the condition $x \le y$ for all $x \in A$ we see that y is an upper bound for A, so by definition of sup, y is at least as large as the supremum of A, that is, $\sup A \le y$.

   (b) Prove that $\sup A \le \inf B$.

   Solution: From part a), $\sup A$ is a lower bound for B, so by definition of inf, $\sup A$ is at least as small as the infimum of B, that is, $\sup A \le \inf B$.

5. A number x is called an *almost upper bound* for A if there are only finitely many numbers $y \in A$ with $y \ge x$; and x is called an *almost lower bound* for A if there are only finitely many numbers $y \in A$ with $y \le x$.

   (a) For each of these sets (that you have already considered in Question 2), find *all* almost upper bounds, and all almost lower bounds.

   i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

   Solution: The set in question is $\{1, 1/2, 1/3, \ldots\}$. Any number strictly greater than 0 is an almost upper bound (and there are no other almost upper bounds). Any number less than or equal to 0 is an almost lower bound (and there are no other almost lower bounds).

   ii. $\{x : x^2 + x + 1 \ge 0\}$

   Solution: The set in question is $\mathbb{R}$, which has no almost upper bounds and no almost lower bounds.

   iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

   Solution: The set in question consists of the sequence of positive numbers

$$\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots$$

   (decreasing, getting ever closer to 1), the sequence of negative numbers

$$\frac{-2}{3}, \frac{-4}{5}, \frac{-6}{7}, \ldots$$

(decreasing, getting ever closer to -1), and also the number 0.
Any number strictly greater than 1 is an almost upper bound (and there are no other almost upper bounds). Any number less than or equal to -1 is an almost lower bound (and there are no other almost lower bounds).

(b) Suppose that A is infinite, and bounded. Prove that the set B of all almost upper bounds of A is non-empty, and bounded from below.

Solution: Since A is bounded, it has at least one upper bound, and that is also an almost upper bound, so the set B of almost upper bounds is non-empty.

Also since A is bounded, it has at least one lower bound, say ℓ. Because A is infinite there are infinitely many elements of A above ℓ, so ℓ is not an almost upper bound, and nor is any number below ℓ, for the same reason. Hence ℓ is a lower bound for the set of all almost upper bounds, and B is indeed bounded from below.

(c) It follows from part (b) that $\inf B$ exists. This number is called the *limit superior* of A, and is denoted by $\limsup A$. For each of the following sets A that are bounded and infinite, find $\limsup A$.

i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Solution: $\limsup A = 0$

ii. $\{x : x^2 + x + 1 \geq 0\}$

Solution: $\limsup A$ does not exist, as set is not bounded from above.

iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

Solution: $\limsup A = 1$

(d) **OPTIONAL**! Define $\liminf A$, and find it for each of these A:

i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

Solution: We define $\liminf A$ to be the supremum of the set of all almost lower bounds of A. By a very similar argument to that given earlier for lim sup, this number exists for all infinite, bounded A.
For $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, $\liminf A = 0$.

ii. $\{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$

Solution: The set of x's in question is the open interval from $(-1 - \sqrt{5})/2$ to 0. The lim inf of this set is $(-1 - \sqrt{5})/2$.

iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

Solution: The lim inf of this set is -1.

6. Remember that a *lower* bound for a set S is a number b such that for all x, if $x \in S$ then $b \leq x$, and a *greatest lower bound* is a lower bound c with the property that if b is any other lower bound, then $b \leq c$. If a set S has a greatest lower bound, then we write it as $\inf S$ ("infimum").

This question shows that the completeness axiom, every non-empty set that has an upper bound, has a least upper bound, ($\star$)

implies the statement

every non-empty set that has a lower bound, has a greatest lower bound ($\star\star$).

The same argument could be used in reverse to show that ($\star\star$) implies ($\star$), so that ($\star\star$) is just an alternative form of the completeness axiom.

(a) Suppose that S is non-empty and has some lower bound. Show that the set $-S$ (meaning, $\{-s : s \in S\}$) is non-empty and has an upper bound.

Solution: Since S is non-empty, there is some $s \in S$. But then $-s \in -S$, so $-S$ is non-empty.

Let b be a lower bound for S. Then $b \leq x$ for all $x \in S$, so $-b \geq -x$ for all $x \in S$. Since every element of $-S$ is of the form $-x$ for some $x \in S$, this shows that $-b \geq y$ for all $y \in -S$, so $-S$ has an upper bound.

(b) Use part (a) and the completeness axiom to show that every non-empty set S that has a lower bound, has a greatest lower bound. **Hint**: Suppose $\alpha = \sup(-S)$. What is a good candidate for $\inf S$?

Solution: If S is non-empty and has a lower bound, then by the previous part $-S$ is non-empty and has an upper bound, so by completeness $-S$ has a least upper bound. Call this α.

We claim that $-\alpha$ is a greatest lower bound for S. First, we show that it is a lower bound. Suppose $x \in S$. Then $-s \in -S$, so $-s \leq \alpha$, so $-\alpha \leq s$. This shows that indeed $-\alpha$ is a lower bound for S.

Next we show that $-\alpha$ is a *greatest* lower bound. Let β be a lower bound for S. Then $\beta \leq x$ for all $x \in S$, so $-x \leq -\beta$ for all $x \in S$. Since every every element of $-S$ is of the form $-x$ for some $x \in S$, this shows that $-\beta$ is an upper bound for $-S$, so $-\alpha \leq -\beta$, or $\beta \leq \alpha$. This shows that indeed $-\alpha$ is a greatest lower bound for S.

7. For this question, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with leading coefficient 1 and with n *even*.

(a) Show that there is a number M such that if $x > M$, then $p(x) > a_0$, and if $x < -M$, then also $p(x) > a_0$.

Solution: If $n = 0$ then the result is automatic — $p(x)$ is the constant function 1. So we can assume $n \geq 2$.

As long as $|x| \geq 1$ (so $|x|^k \leq |x|^\ell$ for $1 \leq k < \ell$) we have

$$\begin{aligned}|a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| &\leq |a_{n-1}||x|^{n-1} + \cdots + |a_1||x| + |a_0| \\ &< (|a_{n-1}| + \cdots + |a_1| + |a_0| + 1)|x|^{n-1} \\ &= L|x|^{n-1}\end{aligned}$$

where $L = |a_{n-1}| + \cdots + |a_1| + |a_0| + 1$.

For $x \geq 1$ we therefore have

$$p(x) > x^n - Lx^{n-1} = x^{n-1}(x - L).$$

We want to show that as long as x is large enough, this expression is at least a_0. If $a_0 \leq 0$, we can simply take $x > L$ to get $p(x) > 0$ (note $L \geq 1$). If $a_0 > 0$, then take $x > \max\{L + 1, a_0 + 1\}$ to get

$$p(x) > (a_0 + 1)^{n-1} > a_0$$

(the last equality following, for example, from the binomial theorem, and using $n \geq 2$). So, regardless of the value of a_0, if $x > \max\{L + 1, a_0 + 1\}$ then $p(x) > a_0$.

For $x \leq -1$ we have

$$p(x) > x^n - L|x|^{n-1} = x^n + Lx^{n-1} = (-x^{n-1})(-x - L).$$

(Note $-x^{n-1}$ is positive). We want to show that as long as x is negative enough, this expression is at least a_0. Let's commit to choosing $x < -L - 1$, so $-x - L > 1$, so it is enough to show $-x^{n-1} > a_0$. If $a_0 \leq 0$, this is instant, since $-x^{n-1}$ is positive. If $a_0 > 0$, then if we also commit to taking $x < -a_0 - 1$ we have

$$-x^{n-1} = (a_0 + 1)^{n-1} > a_0,$$

as before. So we take $x < \min\{-L - 1, -a_0 - 1\}$.

In summary, $M = \max\{L + 1, a_0 + 1\}$ works.

(b) Prove that $p(x)$ is bounded from below and achieves its minimum (i.e., prove that there is a number x_0 such $p(x_0) \leq p(x)$ for all real x). **Note**: because the domain of p is all reals, and not just a closed interval in the reals, you cannot just instantly apply the Extreme Value Theorem to p. You need to use part (a) as well.

Solution: By the Extreme Value Theorem, there is a number x_0 such that $p(x_0) \leq p(x)$ for all $x \in [-M, M]$ (p is continuous on that closed interval).

In part a we found a number M such that for x in the intervals (M, ∞) and $(-\infty, -M)$ we have $p(x) > a_0 = p(0)$.

So for all real x, either $x \in [-M, M]$, in which case $p(x_0) \leq p(x)$ directly from EVT, or $x \in (-\infty, -M) \cup (M, \infty)$, in which case $p(x_0) \geq p(0) > p(x)$ so $p(x_0) > p(x)$, using part (a).