Math 10850, Honors Calculus 1

Homework 8

Due in class Friday November 8

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

This homework covers the Extreme Value Theorem, and upper/lower bounds/functions bounded above/bounded below/completeness. You should read Section 3.7 and 3.8 (on upper/lower bounds/completeness/applications of completeness) and Section 7.4 (Extreme Value Theorem) of the class notes, and/or Spivak Chapter 7 and 8.

Assignment

1. (Note that this question is *not* about applying the Extreme Value Theorem; the given functions may or may not be continuous, and may or may not be defined on closed intervals.)

   For each of the following functions

   (a) say whether they are bounded above, and/or below on the given interval, and

   (b) whether they achieve their maximum and/or minimum value on the given interval.

      i. $f(x) = x^2$ on $(-1, 1)$.

      ii. $f(x) = x^2$ on $[0, \infty)$

      iii. $f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ 1/q & \text{if } x = p/q \text{ in lowest terms, } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ on $[0, 1]$

      iv. $f(x) = \begin{cases} x & \text{if } x \text{ rational} \\ 0 & \text{if } x \text{ irrational} \end{cases}$ on $[0, a]$. Here $a > 0$. The answer may depend on a, so you may need to treat cases.

2. For each the following sets

   (a) find the least upper bound, and the greatest lower bound, if they exist. Note that the l.u.b. and the g.l.b. are *numbers*, so (at least for the purposes of this question) it is not legitimate to say, for example "$\sup A = \infty$".

(b) Also, in the cases where the l.u.b. and/or g.l.b. exists, say whether these values happen to belong to the sets in question.

i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
ii. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$
iii. $\{x : x^2 + x + 1 \geq 0\}$
iv. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

3. **OPTIONAL!** (A little bit of history — this was Archimedes' approach to estimating π)

(a) Suppose that $a_1, a_2, \ldots$ is a sequence of positive numbers with $a_{n+1} \leq a_n/2$. Prove that for any $\varepsilon > 0$ there is some n with $a_n < \varepsilon$. (Here I don't want you to make an assertion like "$1/2^n$ can be made arbitrarily small, by making n sufficiently large", without a clear proof. You may assume the fact that we proved in class, that $\mathbb{N}$ is unbounded.)

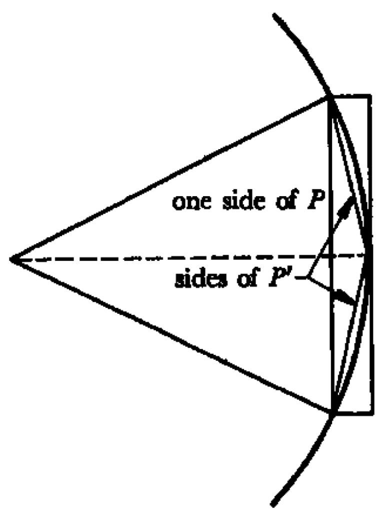
(b) Suppose P is a regular polygon, inscribed inside a circle. If P' is the inscribed regular polygon with twice as many sides as P, show that the quantity

$$\text{area of circle} - \text{area of } P'$$

is less than half the quantity

$$\text{area of circle} - \text{area of } P$$

(see figure below, taken from Spivak Chapter 8).



(c) Show that for every $\varepsilon > 0$, it is possible to inscribe a regular polygon P into a circle, such that the quantity

$$\text{area of circle} - \text{area of } P$$

is less than ε.[1]

[1] Archimedes used this, called the "method of exhaustion", together with an analogous result for *superscribed* polygons, to show $223/71 < \pi < 22/7$.

4. Suppose that A and B are two non-empty sets of numbers such that $x \leq y$ for all $x \in A$ and all $y \in B$.

   (a) Prove that $\sup A \leq y$ for all $y \in B$.

   (b) Prove that $\sup A \leq \inf B$.

5. A number x is called an *almost upper bound* for A if there are only finitely many numbers $y \in A$ with $y \geq x$; and x is called an *almost lower bound* for A if there are only finitely many numbers $y \in A$ with $y \leq x$.

   (a) For each of these sets (that you have already considered in Question 2), find *all* almost upper bounds, and all almost lower bounds.

      i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

      ii. $\{x : x^2 + x + 1 \geq 0\}$

      iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

   (b) Suppose that A is infinite, and bounded. Prove that the set B of all almost upper bounds of A is non-empty, and bounded from below.

   (c) It follows from part (b) that $\inf B$ exists. This number is called the *limit superior* of A, and is denoted by $\limsup A$. For each of the following sets A that are bounded and infinite, find $\limsup A$.

      i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

      ii. $\{x : x^2 + x + 1 \geq 0\}$

      iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

   (d) **OPTIONAL**! Define $\liminf A$, and find it for each of these A:

      i. $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

      ii. $\{x : x < 0 \text{ and } x^2 + x - 1 < 0\}$

      iii. $\left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$

6. Remember that a *lower* bound for a set S is a number b such that for all x, if $x \in S$ then $b \leq x$, and a *greatest lower bound* is a lower bound c with the property that if b is any other lower bound, then $b \leq c$. If a set S has a greatest lower bound, then we write it as $\inf S$ ("infimum").

   This question shows that the completeness axiom,

   every non-empty set that has an upper bound, has a least upper bound, $(\star)$

   implies the statement

   every non-empty set that has a lower bound, has a greatest lower bound $(\star\star)$.

   The same argument could be used in reverse to show that $(\star\star)$ implies $(\star)$, so that $(\star\star)$ is just an alternative form of the completeness axiom.

(a) Suppose that S is non-empty and has some lower bound. Show that the set $-S$ (meaning, $\{-s : s \in S\}$) is non-empty and has an upper bound.

(b) Use part (a) and the completeness axiom to show that every non-empty set S that has a lower bound, has a greatest lower bound. **Hint**: Suppose $\alpha = \sup(-S)$. What is a good candidate for $\inf S$?

7. For this question, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial with leading coefficient 1 and with n *even.*

   (a) Show that there is a number M such that if $x > M$, then $p(x) > a_0$, and if $x < -M$, then also $p(x) > a_0$.

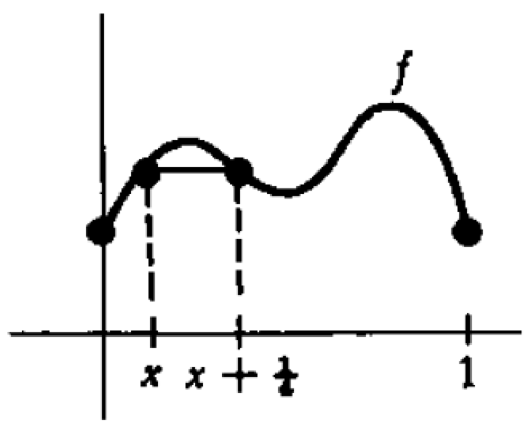
   (b) Prove that $p(x)$ is bounded from below and achieves its minimum (i.e., prove that there is a number x_0 such $p(x_0) \leq p(x)$ for all real x). **Note**: because the domain of p is all reals, and not just a closed interval in the reals, you cannot just instantly apply the Extreme Value Theorem to p. You need to use part (a) as well.

Extra credit problem

A two-part problem, possible quite hard:

1. (Easier) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, with $f(0) = f(1)$. Let $a = 1/n$, where n is a natural number.

   Prove that there is some number x such that $f(x) = f(x + a)$. (The figure below, taken from Spivak Chapter 7, gives an illustration for $n = 4$.)



2. (Harder) For each number a in $(0, 1)$ that is *not* of the form $1/n$ for some natural number n, find a continuous function $f_a : [0, 1] \to \mathbb{R}$ with $f_a(0) = f_a(1)$ but which there is *no* number x with $f_a(x) = f_a(x + a)$.