Math 10850, Honors Calculus 1

Homework 4

Due in class Friday September 27

General and specific notes on the homework

All the notes from homework 1 still apply!

Reading for this homework

Section 4 of course notes (Sections 4.1 through 4.5 are the important sections; the remaining sections are for your general edification). The material is covered in Chapter 2 of Spivak.

Assignment

1. Prove the following identities. The main point here is that you should be working towards laying out your proof in a clear and organized manner. Use the proof from class that $1 + 2 + \ldots + n = n(n+1)/2$ as a template.

   - Begin the proof by saying that it will be a proof by induction on n.
   - Verify the base case, and when you do so, clearly indicate that that is what you are doing
   - When you move onto to the induction step, clearly indicate that that is what you are doing.
   - In the induction step, explicitly state what you are assuming (the inductive hypothesis), and then clearly deduce what you want to deduce.
   - End with a concluding statement, along the lines of "By induction, we conclude that ...".)

   (a) For all natural numbers n,

   $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

   Note that this says: the sum of the *cubes* of the first n numbers, is the same as the *square* of the sum of the first n numbers; an odd fact!

(b) Remember that the Fibonacci numbers are defined by the recurrence relation

$$f_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f_{n-1} + f_{n-2} & \text{if } n \geq 2. \end{cases}$$

Prove that for all $n \geq 0$,

$$\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}.$$

(c) For all natural numbers n,

$$\sum_{k=1}^{n} (3k^2 - 3k + 1) = ???.$$

(Here I'll leave it up to you to find the correct right-hand side — a simple expression that doesn't involve a sum — and then prove that what you have found is correct)

2. (a) Let r be a real number that's not equal to 1. Prove by induction on n that

$$1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

(b) Set

$$S = 1 + r + r^2 + \ldots + r^n.$$

By multiplying both sides by r and doing some algebraic manipulation on the two equations, give a different (non-induction) proof of the result from the part (a).

3. In class we defined the expression a^n for all real a and all natural numbers n, via a recursive definition. Prove (by induction) that for all natural numbers n and m we have

$$a^{n+m} = a^n a^m.$$

(Hint: don't try to be too fancy with the induction; pick either induction on n or induction on m, but not both at once.)

4. Prove that if p, q are rational numbers, $x = p + \sqrt{q}$, and m is a natural number, then $x^m = a + b\sqrt{q}$ for some rational numbers a, b.

5. Identify[1] the error in the following proof of the claim "All cows are the same color":

Let $p(n)$ be the predicate "any n cows are the same color". We prove that $p(n)$ is true for all $n \geq 1$ (and so that all cows are the same color), by induction on n.

Base case $n = 1$: any one cow is a set of cows all of which are the same color (whatever color the cow under consideration is). So $p(1)$ is true.

[1] *Clearly* identify the *specific* error — vagueness not acceptable here!

Induction step: Suppose that for some $n \geq 1$, $p(n)$ is true. Let

$$\{\text{Cow}_1, \text{Cow}_2, \ldots, \text{Cow}_n, \text{Cow}_{n+1}\}$$

be a set of $n+1$ cows. By the induction hypothesis (the fact that $p(n)$ is True), all of Cow_1, Cow_2, …, Cow_n are the same color; call that color C. Also by the induction hypothesis, all of Cow_2, Cow_3, …, Cow_n, Cow_{n+1} are the same color (this is another collection of n cows). That common color must be C, because Cow_2 (for example) is colored C, from the first application of induction hypothesis. It follows that all of Cow_1, Cow_2, …, Cow_n, Cow_{n+1} are the same color, C, and so $p(n+1)$ is True.

By induction, we conclude that $p(n)$ is True for all $n \geq 1$, and so all cows are the same color.

6. The Fibonacci numbers (defined in question 1) are very closely related to the *golden ratio*, the number $(1+\sqrt{5})/2 \approx 1.618$, that is often denoted φ.

   (a) Prove (most easily by induction on n) that for $n \geq 1$,

   $$f_n \leq \varphi^{n-1}.$$

   (Be careful! There's a slight trap in this question, into which you may fall if you are not careful.)

   (b) Prove that that for $n \geq 1$

   $$f_n \geq \varphi^{n-2}.$$

   Note: These two parts together show that f_n grows roughly at the same rate as φ^n; specifically, for all $n \geq 1$

   $$0.3819 \approx \frac{1}{\varphi^2} \leq \frac{f_n}{\varphi^n} \leq \frac{1}{\varphi} \approx 0.6180.$$

   It's possible to be more precise, and show that for all large n

   $$\frac{f_n}{\varphi^n} \approx \frac{1}{\sqrt{5}} \approx 0.4472.$$

   (And it's possible to be *much* more precise, and give an exact formula for f_n in terms of φ).

7. Prove that for all natural numbers n, the expression

   $$2 \times 7^n + 3 \times 5^n - 5$$

   is divisible by 24. (It will be helpful to know that if a divides b, and a divides c, then a divides any linear combination of b and c; that is, a divides $mb + nc$ for every pair of integers m, n).

8. Prove the generalized triangle inequality: for all natural numbers n, if $x_1, x_2, \ldots, x_n$ are real numbers, then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

9. For whole numbers $n \geq \ell \geq 0$ let

$$f(n, \ell) = \sum_{k=0}^{\ell} (-1)^k \binom{n}{k}$$

(so $f(n, \ell)$ is the alternating sum of the entries along the nth row of Pascal's triangle, up to and including the term $\binom{n}{\ell}$). For example

- $f(0, 0) = (-1)^0\binom{0}{0} = 1$,
- $f(1, 0) = (-1)^0\binom{1}{0} = 1$,
- $f(1, 1) = (-1)^0\binom{1}{0} + (-1)^1\binom{1}{1} = 0$,
- $f(2, 0) = (-1)^0\binom{2}{0} = 1$,
- $f(2, 1) = (-1)^0\binom{2}{0} + (-1)^1\binom{2}{1} = -1$,
- $f(2, 2) = (-1)^0\binom{2}{0} + (-1)^1\binom{2}{1} + (-1)^2\binom{2}{2} = 0$, and
- $f(5, 3) = (-1)^0\binom{5}{0} + (-1)^1\binom{5}{1} + (-1)^2\binom{5}{2} + (-1)^3\binom{5}{3} = -4$.

By computing $f(n, \ell)$ for a bunch more small values of n and ℓ (by hand, or by computer), conjecture a simple formula for $f(n, \ell)$ and prove that the formula is correct.

10. In class we saw that the general associative law — no matter how parentheses are placed around the expression $a_1 + a_2 + \ldots + a_n$, the sum is still the same — follows from the associativity axiom.

    Show that the general commutative law — no matter what order $a_1, a_2, \ldots, a_n$ are added in, the sum is still the same — follows from the commutativity axiom $a+b = b+a$. You may assume the general associative law.

    (As a specific clarifying example, the case $n = 3$ of the general commutative law says that $a + b + c$, $a + c + b$, $b + a + c$, $b + c + a$, $c + a + b$ and $c + b + a$ are all the same.)

An extra credit problem

Please submit this on a *separate* sheet.

On an infinite sheet of white graph paper (a paper with a square grid), n squares are colored black. At moments $t = 1, 2, \ldots$, squares are recolored according to the following rule: each square gets the color occurring at least twice in the triple formed by this square, its top neighbor, and its right neighbor.

1. Prove that after the moment $t = n$, all squares are white.

2. Can you find, for infinitely many n, an initial configuration of n squares such that *before* the moment $t = n$ there are still some squares colored black?