Math 10850, Honors Calculus 1

Homework 3

Due in class Friday September 20

General and specific notes on the homework

All the notes from homework 1 still apply!

This homework is **not** about verifying everything in gory, graphic detail, directly from the axioms. Rather, it is about using the ideas and concepts that we are seeing, to do actual math.

- Some of the questions (2, 5, 7) are purely computational; although these aren't "proofs" per se, you still need to convey clearly what substantial steps you are taking to solve the problems, and you should still be aiming to present your answers in coherent and complete sentences.
- Others (1, 4, 6) are slightly more conceptual explorations of things that we have seen in class, which again require coherent and complete answers, that are intelligible to a grader who is reading your work, and who doesn't have the chance to ask you for clarification.
- One (3) involves proofs, which should be presented as a narrative with complete sentences. You don't need to justify obvious steps (like routine applications of the axioms), and you should feel comfortable taking multiple steps at once (like writing $(x-y)^2 = x^2 - 2xy + y^2$ without any further fuss). Non-obvious steps should be explained, though.

The homework concerns absolute value, inequalities, and irrational numbers.

Reading for this homework

Sections 3.5, 3.6 and 3.7 of the course notes (though there are no questions on Section 3.7).

Assignment

1. Express each of the following without absolute value signs, treating various cases separately where necessary. Try to use as few cases as possible. Write your final solution using the *brace notation*, for example

$$\text{this thing} = \begin{cases} \text{something} & \text{if condition/case 1} \\ \text{something else} & \text{if condition 2} \\ \text{something else again} & \text{if condition 3} \end{cases}$$

(a) $|a+b|-|b|$ (where a, b might be any real numbers).

(b) $a-|(a-|a|)|$ (where a might be any real number).

2. Find all real numbers x for which:

   (a) $|x+4|<2$.

   (b) $|x-1|+|x+1|<2$.

   (c) $|x-1| \cdot |x+2| = 3$.

3. Prove each of the following inequalities (if you think it might be useful, you can assume the triangle inequality):

   (a) $|x-y| \leq |x|+|y|$ for all reals x, y.

   (b) $|x|-|y| \leq |x-y|$ for all reals x, y.

4. The maximum of two numbers x, y is denoted $\max\{x, y\}$, and the minimum is denoted $\min\{x, y\}$. So, for example,

   - $\max\{3, -1\} = 3$
   - $\min\{4.5, 4.5\} = 4.5$
   - $\min\{-3, -4\} = -4$.

   (a) Prove that $\max\{x, y\} = \frac{x+y+|y-x|}{2}$.

   (b) Find a very similar formula for $\min\{x, y\}$.

   (c) Find a formula for $\max\{x, y, z\}$ (the maximum of the three numbers x, y and z). You can use x, y, z, addition, subtraction, division, multiplication, and absolute value, but **not** max or min.

   (d) Define $\mathrm{middle}\{x, y, z\}$ to be the middle number of x, y and z when the three are written in increasing order (so, for example, $\mathrm{middle}\{7, -8, 2\}$ is 2, and $\mathrm{middle}\{0, 0, 1\}$ is 0). Find a formula for $\mathrm{middle}\{x, y, z\}$ that only uses x, y, z, addition, subtraction, division, multiplication, and absolute value.

5. Although it is not immediately apparent, this question is related to the fact that if $a \neq 0$ then $a^2 > 0$ (that's a hint).

   (a) Find the smallest possible value of $2x^2-3x+4$ as x runs over real numbers.

   (b) Find the smallest possible value of $x^2-3x+2y^2+4y+2$ as x and y run over real numbers.

   (c) Find the smallest possible value of $x^2+4xy+5y^2-4x-6y+7$ as x and y run over real numbers.

6. (Here, if your answer to a part is "yes" then you need to give a proof; if your answer is "no" you need to give an example that illustrates this. A *rational* number is a number that can be expressed as the ratio of two whole numbers. A number which is not rational is *irrational*. Remember that in class we proved that $\sqrt{2}$ is a real number that is irrational.)

   (a) If a is rational and b irrational, is $a+b$ necessarily irrational?

   (b) If a is irrational and b irrational, is $a+b$ necessarily irrational?

   (c) If a is rational and b irrational, is ab is necessarily irrational?

   (d) Is there a number a such that a^2 is irrational but a^4 is rational?

   (e) Does there exist two irrational numbers whose sum and product are both rational?

7. (a) In class we used a "parity trick" (looking at oddness and evenness of a proposed numerator and denominator) to prove that $\sqrt{2}$ is not rational. Use a variant of this trick to prove that $\sqrt{6}$ is irrational. (One possibility is to think about remainder on division by 6).

   (b) Prove that $\sqrt[3]{2}$ is irrational.

   (c) Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

An extra credit problem

Please submit this on a *separate* sheet.

This problem follows on from Question 4. Given n numbers $x_1, x_2, \ldots, x_n$, define

$$\mathrm{kth} - \mathrm{largest}\{x_1, x_2, \ldots, x_n\}$$

to be the kth largest of the n numbers. So, for example:

- $\mathrm{4th} - \mathrm{largest}\{2, -1, 0, 2, 8, 9, 5, 2, 6, 4\} = 5$
- $\mathrm{nth} - \mathrm{largest}\{x_1, x_2, \ldots, x_n\}$ is the minimum of $x_1, x_2, \ldots, x_n$
- $\mathrm{1th} - \mathrm{largest}\{x_1, x_2, \ldots, x_n\}$ is the maximum of $x_1, x_2, \ldots, x_n$.

Show that is is possible, for every n and k, to compute $\mathrm{kth} - \mathrm{largest}\{x_1, x_2, \ldots, x_n\}$ using only the numbers $x_1, x_2, \ldots, x_n$ and k, and the operations of addition, subtraction, division and taking absolute values.

(You are unlikely to actually come up with an explicit expression — it would be absolutely hideous — so instead you should aim to explain the process, or algorithm, you would implement in order to do the computation.)