Math 10850, Honors Calculus 1

Homework 2

Solutions

1. Begin by re-reading the guidelines for Homework 1, and by reading either Spivak, Chapter 1, or the suggested reading from the course notes for this homework.

2. Using only the axioms P1 through P12, the closure of the set of numbers under addition and multiplication, and the assumption $0 \neq 1$, prove each of the following. Justify *every* line of your proof.

   (a) The *cancellation* property of addition: for all real numbers a, b, c, if $a + b = a + c$ then $b = c$.

   Solution: Since $a + b = a + c$ we have $-a + (a + b) = -a + (a + c)$ ($-a$ exists by P3), so $(-a + a) + b = (-a + a) + c$ (P1), so $0 + b = 0 + c$ (P3), so $b = c$ (P2).

   (b) The uniqueness of the multiplicative inverse, in a strong form: if a and x are numbers satisfying $ax = a$ and $a \neq 0$ then $x = 1$. (**Comment**: this implies that if x is a multiplicative inverse, that is, if $ax = xa = a$ for all $a \neq 0$, then $x = 1$).

   Solution: If $ax = a$ and $a \neq 0$, then $a^{-1}(ax) = a^{-1}a = 1$ (P7 for existence of a^{-1}, then P7 again for second equality), so $(a^{-1}a)x = 1$ (P5), so $1x = 1$ (P7), so $x = 1$ (P6).

   (c) If $x^2 = y^2$ then either $x = y$ or $x = -y$.

   Solution: Here I'm not going to explain which axiom is being used in every line. Also, I'm sneaking in a result from class — $-(ab) = (-a)b$.
   If $x^2 = y^2$ then $x^2 + (-y^2) = y^2 + (-y^2) = 0$, so (by definition of subtraction) $x^2 - y^2 = 0$. Now

$$\begin{aligned}
(x-y)(x+y) &= (x-y)x + (x-y)y \\
&= (x^2 - yx) + (xy - y^2) \\
&= (x^2 - yx) + (yx - y^2) \\
&= ((x^2 - yx) + yx) - y^2 \\
&= (x^2 + (-yx + yx)) - y^2 \\
&= (x^2 + 0) - y^2 \\
&= x^2 - y^2.
\end{aligned}$$

   So if $x^2 = y^2$ then $(x - y)(x + y) = 0$, which implies (by a result we proved in class) that either $x - y = 0$ (so $x = y$) or $x + y = 0$ (so $x = -y$).

(d) For all $n \geq 2$, $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$. (Here you can use the extra facts, not yet proven, that the order of terms in a sum or product doesn't matter, and that the order of parenthesizing a sum or product doesn't matter. You don't have to be overly precise here. Just do enough to convince yourself

A that the identity is true, and

B that you could, if forced, and if given enough time, prove it directly from the axioms for any particular n.

Then remember the identity — it's very useful.)

Solution: Distributivity tells us that $a(b+c) = ab + ac$. What about $a(b+c+d)$? Well,

$$a(b+c+d) = a((b+c)+d) = a(b+c) + ad = (ab+ac) + ad = ab + ac + ad.$$

Similarly, we could show (and need to for this problem) that

$$a(a_1 + a_2 + \cdots + a_n) = aa_1 + aa_2 + \cdots + aa_n.$$

This last statement was rather vague. Soon we will see the method of induction, that will allow us to do this kind of general computation quite rigorously; for the moment we will settle for saying that we could, if pressed (and given enough time and paper), verify this for any particular n.

Now we can say $(x-y)(x^{n-1} + yx^{n-2} + \ldots xy^{n-2} + y^{n-2})$

$$\begin{aligned}
&= (x-y)x^{n-1} + (x-y)yx^{n-2} + \ldots + (x-y)xy^{n-2} + (x-y)y^{n-1}) \\
&= x^n - yx^{n-1} + xyx^{n-2} - y^2x^{n-2} + \ldots + x^2y^{n-2} - yxy^{n-2} + xy^{n-1} - y^n \\
&= x^n - yx^{n-1} + yx^{n-1} - y^2x^{n-2} + \ldots + y^{n-2}x^2 - y^{n-1}x + y^{n-1}x - y^n.
\end{aligned}$$

In this last line I have taken a few steps at one time, using commutativity and associativity of multiplication a few times inside each term being added up, to gather x's and y's together, and put the y's to the left.

Now notice that all the terms (except the first and last) pair off into pairs that are additive inverses of each other — the second and third term, the fourth and fifth, etc. These terms add (pairwise) to 0, and all these 0's add to 0, so the entire sum collapses to $x^n - y^n$; this verifies the claimed identity.

3. What is wrong with the following proof that $2 = 1$? (Where of course "2" means "$1 + 1$".)

> Let x be any number, and let $y = x$, so that $x^2 = xy$ and $x^2 - y^2 = xy - y^2$. Factorizing both sides, $(x+y)(x-y) = y(x-y)$ and so $x + y = y$. Using $y = x$ this says $2y = y$ so $2 = 1$.

Solution: In going from line 3 to line 4 we have divided by $x - y$ (or, equivalently, multiplied by the multiplicative inverse of $x - y$). But $x = y$ so $x - y = 0$ and 0 does not have an inverse.

4. In this question, you must use *only* the axioms P1 through P12, the closure of the set of numbers under addition and multiplication, and the assumption $0 \neq 1$, together with the definitions of $<$ and $>$. I expect you to verify each property *very carefully*. In the world of inequalities, many facts are far from obvious (why does multiplying by a negative change the direction of the inequality, for example?), so I want you to leave this question absolutely convinced that the order axioms really are all that is needed to create a notion of $<$ and $>$ that agrees with our intuitive understanding.

   After you have started proving some of the properties asked for in the question, you can use those to establish later properties, if it seems appropriate.

   (a) If $a < b$ and $c < d$ then $a + c < b + d$.

   Solution: If $a < b$ and $c < d$ then, by definition, $b - a$ and $d - c$ are both positive, so their sum is also positive. Using associativity and commutativity of addition we get that the sum is $(b + d) - a - c$. But

$$-a - c = -(a + c),$$

   as can be seen by adding $-a - c$ to $a + c$ to get 0, and appealing to uniqueness of inverse. So we get that $(b + d) - (a + c)$ is positive, from which we conclude that $a + c < b + d$.

   (b) If $a < b$ then $-b < -a$.

   Solution: If $a < b$ then $b - a$ is positive. To establish $-b < -a$ it suffices to show $-a - (-b)$ is positive. So if we show $b - a = -a - (-b)$, we are done. To verify this last, we unwrap the definition of subtraction — $b - a$ means $b + (-a)$, and $-a - (-b)$ means $(-a) + (-(-b))$. Now note that $-(-b) = b$, exactly because $b + (-b) = 0$ (this says both that $-b$ is the additive inverse of b, and that b is the additive inverse of $-b$, that is, that $-(-b) = b$). So $b - a = -a - (-b)$ is equivalent to $b + (-a) = (-a) + b$, which is clearly true.

   (c) If $a < b$ and $c < 0$ then $ac > bc$.

   Solution: If $a < b$ then $b - a$ is positive, and if $c < 0$ then $0 - c = -c$ is also positive. By multiplicative closure $(b - a)(-c) = b(-c) - a(-c)$ is positive. Now $b(-c) = -(bc)$ (we proved in class that $(-c)b = -(cb)$; that $b(-c) = -(bc)$ follows from commutativity), and similarly $a(-c) = -(ac)$, so $-a(-c) = -(-(ac)) = ac$ (using something we derived in the solution to part (ii)). It follows that $b(-c) - a(-c) = -(bc) + ac = ac - bc$, and the positivity of this shows that $ac > bc$.

   (d) If $0 \leq a < b$ and $0 \leq c < d$ then $ac < bd$.

   Solution: If $a = 0$ then $ac = 0$. Since $b > 0$ (so b is positive) and $d > 0$ (so d is positive) we have bd positive, so $bd > 0 = ac$. Similarly $bd > ac$ if $c = 0$. So we may assume that $b > a > 0$ and $d > c > 0$.

Now we know that $bd > bc$ (since $d > c$ and $b > 0$), and we also know that $bc > ac$ (since $b > a$ and $c > 0$). From these two inequalities we conclude $bd > ac$.

(If you want to fully justify this last step: $x > y$ means $x - y$ positive; $y > z$ means $y - z$ positive; additive closure then says $(x - y) + (y - z) = x - z$ positive, so $z > x$.)

Alternate (and better) proof: As before, we may assume $b > a > 0$ and $d > c > 0$. This tells us that all of

$$a,\ b,\ c,\ d,\ b - a,\ d - c$$

are positive. So by P11, P12, we have

$$b(d - c) + c(b - a)$$

is positive. But

$$b(d - c) + c(b - a) = bd - ac,$$

so indeed $ac < bd$.

(e) If $a, b \geq 0$ and $a^2 < b^2$ then $a < b$.

Solution: If $b = 0$ then the condition $a^2 < b^2$ becomes $a^2 < 0$, and there is no valid a satisfying this (0 squares to 0, and all other a square to something positive). So we may assume $b > 0$. If $a = 0$ then the condition $a^2 < b^2$ becomes $0 < b^2$, and we want to conclude $0 < b$, which is true (since we have ruled out $b = 0$).

So now we assume $a, b > 0$. We may assume $a \neq b$, for if $a = b$ then $a^2 = b^2$ and we cannot have the hypothesis $a^2 < b^2$ hold. From $a^2 < b^2$ we get $b^2 - a^2 > 0$ so $(b - a)(b + a) > 0$. Since $a, b > 0$ we have $b + a > 0$. Now suppose $b - a < 0$. In this case, from part c) above, we get $(b - a)(b + a) < (b - a)0 = 0$, a contradiction. And we've already ruled out $b - a = 0$, so we conclude $b - a > 0$ or $a < b$, as required.

5. Prove that if x and y are not both 0, then

(a) $x^2 + xy + y^2 > 0$.

Solution: If $x = y$ then $x^2 + xy + y^2 = 3x^2$, which is positive since $x \neq 0$. So from here on we assume $x \neq y$. Also, if either $x = 0$ or $y = 0$ then the claim is very easy (the expression $x^2 + xy + y^2$ collapses to either x^2 or y^2), so we also assume $x, y \neq 0$.

We know from question 1 that

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

If $x > y > 0$ then $x^3 > y^3 > 0$ (by basic properties of inequalities), so all of $x - y$, $(x - y)^{-1}$ and $x^3 - y^3$ are positive, so

$$(x - y)^{-1}(x^3 - y^3) = x^2 + xy + y^2$$

is positive, as required.

If $y > x > 0$ then $y^3 > x^3 > 0$, so all of $x - y$, $(x-y)^{-1}$ and $x^3 - y^3$ are negative, so

$$(x-y)^{-1}(x^3 - y^3) = x^2 + xy + y^2$$

is positive, as required.

What if $x > 0 > y$? In this case

$$x^2 + xy + y^2 = x^2 - x|y| + |y|^2, \quad x - y = x + |y| \quad \text{and} \quad x^3 - y^3 = x^3 + |y|^3,$$

and we have the factorization

$$x^3 + |y|^3 = (x + |y|)(x^2 - x|y| + |y|^2).$$

Since $x^3 + |y|^3$ and $(x + |y|)$ are both positive, so also is $x^2 - x|y| + |y|^2 = x^2 + xy + y^2$.

The final case, $y > 0 > x$, is almost identical to the last one considered. Details omitted.

(b) $x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0$.

Solution: This is almost identical to $x^2 + xy + y^2$, simply replacing

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

with

$$x^5 - y^5 = (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

I omit the details.

6. (a) Show that $(x+y)^2 = x^2 + y^2$ only if[1] $x = 0$ or $y = 0$.

Solution: We have $(x+y)^2 = x^2 + 2xy + y^2$, so $(x+y)^2 = x^2 + y^2$ is equivalent to $2xy = 0$, which is equivalent to $xy = 0$, which (as we have shown in class) indeed implies $x = 0$ or $y = 0$.

(b) Show that $(x+y)^3 = x^3 + y^3$ only if $x = 0$ or $y = 0$ or $x = -y$.

Solution: We have $(x+y)^3 = x^2 + 3x^2y + 3xy^2 + y^3$, so $(x+y)^3 = x^3 + y^3$ is equivalent to $3x^2y + 3xy^2 = 0$, which is equivalent to $3xy(x+y) = 0$, which (easily extending a result we showed in class) indeed implies $x = 0$, $y = 0$ or $x = -y$.

[1] What does "only if" mean? "p only if q" means that if q doesn't happen, then neither does p, i.e., "(not q) implies (not p)", which is the contrapositive of (and so equivalent to) "p implies q". So in this particular question, you being asked to show that **if** $(x+y)^2 = x^2 + y^2$ **then** either $x = 0$ or $y = 0$; not the other direction (which is somewhat trivial).

Note that this explains the language "if and only if": when we say "p if and only if q", we are saying "p only if q" — i.e., "p implies q" — *and* "p if q" — i.e., "q implies p".

(c) Using the fact that $(x+y)^2$ is not negative, show that $4x^2+6xy+4y^2>0$ unless[2] x and y are both 0.

Solution: We have

$$4x^2+6xy+4y^2 = 3(x^2+2xy+y^2)+x^2+y^2 = 3(x+y)^2+x^2+y^2.$$

The right-hand side is the sum of three non-negative numbers, so is non-negative, and the only way that it can be 0 is if each of the three summands is 0, which only happens if both x and y are 0; so as long as x and y are not 0, $4x^2+6xy+4y^2>0$.

(d) Use part (c) to find all x, y for which $(x+y)^4 = x^4+y^4$.

Solution: We have

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4,$$

so $(x+y)^4 = x^4+y^4$ is equivalent to $4x^3y+6x^2y^2+4xy^3=0$, which is equivalent to $xy(4x^2+6xy+4y^2)=0$. This certainly happens if either $x=0$ or $y=0$, but if neither $x=0$ nor $y=0$ then, since $4x^2+6xy+4y^2>0$ in this case (by part (b)), we do not have $xy(4x^2+6xy+4y^2)=0$. Hence $(x+y)^4=x^4+y^4$ only if $x=0$ or $y=0$.

7. This question and the next question are concerned with the *absolute value* function. We may not get to this in class on Wednesday, but that's ok — I'll define it here, and you'll get the chance to think about it before we see it in class... (and/or you can look at Section 3.6 of the course notes).

The *absolute value* of a real number x, denoted $|x|$ is defined to be

$$x, \text{ if } x \geq 0$$

and

$$-x, \text{ if } x < 0.$$

So, x is the distance from x to 0 on the number line (always a non-negative number).

Let ε be a positive number. Prove that if $|x-x_0|<\varepsilon/2$ and $|y-y_0|<\varepsilon/2$ then both of

$$|(x+y)-(x_0+y_0)|<\varepsilon$$

and

$$|(x-y)-(x_0-y_0)|<\varepsilon$$

[2] What does "unless" mean? "p unless q" means that the only way for p not to happen, is for q to happen, that is, "(not p) only if q". As we've seen in the previous footnote, this is the same as "(not p) implies q". So in this particular question, you being asked to show that **if** $4x^2+6xy+4y^2 \leq 0$ **then** both $x=0$ and $y=0$. The contrapositive of this, which might be easier to think about, is that **if** at least one of x, y are not 0 then $4x^2+6xy+4y^2>0$.

hold. (Not an idle question; we will need these kinds of manipulations when we come to study limits and continuity.)

Solution: If $|x - x_0| < \varepsilon/2$ and $|y - y_0| < \varepsilon/2$, then we have

$$\begin{array}{rcccl} -\varepsilon/2 & < & x - x_0 & < & \varepsilon/2 \\ -\varepsilon/2 & < & y - y_0 & < & \varepsilon/2. \end{array}$$

Adding these two rows of inequalities we get

$$-\varepsilon < (x + y) - (x_0 + y_0) < \varepsilon$$

or $|(x + y) - (x_0 + y_0)| < \varepsilon$; subtracting the second from the first we get

$$-\varepsilon < (x - y) - (x_0 - y_0) < \varepsilon$$

or $|(x - y) - (x_0 - y_0)| < \varepsilon$.

8. Prove the *reverse triangle inequality*: for all numbers a, b,

$$||a| - |b|| \leq |a - b|.$$

Also, figure out for which choices of a and b there is equality.

(**Hint**: Break your analysis into cases, choosing the cases in such a way that inside each case, you can remove all the absolute values from both sides of the inequality that you are trying to prove.)

Solution: Here is a solution based on the direct method for proving the triangle inequality given by Spivak:

Both sides are non-negative, so the fact that $x \leq y$ if and only if $x^2 \leq y^2$ for $x, y \geq 0$ means that it is enough to verify

$$(||a| - |b||)^2 \leq (|a - b|)^2.$$

Now

$$(||a| - |b||)^2 = |a|^2 - 2|a||b| + |b^2| = a^2 - 2|a||b| + b^2$$

and

$$(|a - b|)^2 = (a - b)^2 = a^2 - 2ab + b^2.$$

So we need to show

$$a^2 - 2|a||b| + b^2 \leq a^2 - 2ab + b^2$$

or

$$-2|a||b| \leq -2ab$$

or

$$ab \leq |a||b|,$$

which is true (it was the key point in the proof of the ordinary triangle inequality).

If $||a| - |b|| = |a - b|$ then, squaring both sides and rearranging, we get $ab = |a||b|$, which happens if either of a or b is 0, or if a and b are either both positive or both negative; and it is easy to check that there is indeed equality in all of these cases. (Note that these are exactly the cases of equality in the ordinary triangle inequality)

Here is a proof based on the case analysis proof given for the triangle inequality in class:

We consider cases.

If $a = 0$ then the inequality to be proven becomes $|0 - |b|| \leq |0 - b|$ or $|b| \leq |b|$, which is true, with equality, for all b. Similarly if $b = 0$ then the inequality holds with equality for all a.

If a and b are both positive then the inequality becomes $|a - b| \leq |a - b|$, which is true with equality for all a, b. Similarly if a, b are both negative then it becomes $|-a-(-b)| \leq |a - b|$, or $|b - a| \leq |a - b|$, which is true with equality for all a, b (note that for all x, $|-x| = |x|$, the common value being x if x is at least 0 and $-x$ if x is at most 0).

If a is positive and b is negative then the inequality becomes $|a + b| \leq |a - b|$. Since $-b$ is positive, this is the same as $|a + b| \leq a - b$. Now $|a + b|$ is either $a + b$ or $-a - b$. If it is $a + b$, then the inequality to be proven becomes $a + b \leq a - b$ or $b \leq -b$, which holds with *strict* inequality when b is negative; and if it is $-a - b$, then the inequality to be proven becomes $-a - b \leq a - b$ or $-a \leq a$, which holds with *strict* inequality when a is positive. So in this case the inequality holds, with strict inequality.

For the final case, a negative and b positive, a similar analysis to the previous case shows that again the inequality holds strictly.

In conclusion, the inequality is true for all a, b, and there is equality if either of a or b is 0, or if a and b are either both positive or both negative.

9. In class we used the distributive axiom P9 to show that $a \cdot 0 = 0$ for all real numbers a.

   (a) Show that it is *necessary* to use P9 (or something like it) to prove this. More specifically: find a set of "numbers", that includes special "numbers" "0" and "1" (different from each other), for which there is a notion of "addition" and "multiplication", that satisfies all of the axioms P1 through P9, but for which it is *not* the case that $a \cdot 0 = 0$ for all numbers a.

   Solution: The key here is that axioms P1 through P4 only mention addition, and axioms P5 through P9 only mention multiplication; there is no connection between the two operations, without P9. So if you take essentially *any* set of numbers with essentially *any* operation of addition, that satisfies P1 through P4, and you take the same set set of numbers with essentially *any* operation of multiplication, that satisfies P5 through P8, then there is a very good chance that the way that they interact with each other is *not* the way that is specified by P9 (why should it be? that's one, very specific, mode of interaction, that isn't likely to happen by chance).

   Here is possible the simplest example. The set of numbers is $\{0, 1\}$, and addition is defined by

- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 0$.

Notice that this is exactly the addition that we defined in class, in our example that showed that P1 through P9 is not enough to prove that $1 + 1 \neq 0$ — it is the even/odd addition.

Multiplication is defined by

- $0 \cdot 0 = 1$
- $0 \cdot 1 = 0$
- $1 \cdot 0 = 0$
- $1 \cdot 1 = 1$.

If you think about it, you'll see that this is just the even/odd addition table, now with "even" being identified with "1" and "odd" with "0", instead of vice-versa — I did this to make sure that the multiplicative identity is a different number to the additive identity.

It is an easy (but somewhat tedious) check that this system satisfies all of P1 through P8. But $0 \cdot 0 = 1$, so it is *not* the case that $a \cdot 0 = 0$ for all a.

(b) P9 must fail in the system you discovered in the previous part (otherwise, you could prove that $a \cdot 0$ always equal 0, using the proof we had in class). Give an explicit example of the failure of the distributive axiom in your system.

Solution: We have

$$0 \cdot (1 + 1) = 0 \cdot 0 = 1,$$

while

$$0 \cdot 1 + 0 \cdot 1 = 1 + 1 = 0.$$

So P9 fails:

$$0 \cdot (1 + 1) \neq 0 \cdot 1 + 0 \cdot 1.$$