Math 10850, Honors Calculus 1

Homework 2

Due in class Friday September 13

General and specific notes on the homework

All the notes from homework 1 still apply!

The questions here concern the real numbers, *as we have defined them* — a set, together with special numbers $0, 1$, and with operations of addition and multiplication, that satisfy the axioms P1 through P13 (though P13 will not appear on this homework). When working on these problems, you are of course at liberty to use your intuitive understanding of the reals to guide you; but you **must** present your proofs in the context of the axioms: every step you take must *either* be valid from an axiom, *or* from a result we have have previously established in class, *or* from a result you have previously established in the homework.

That being said, I *don't* expect you to laboriously label each step. Except in questions 2 (a through c) and 4, where specific instructions are given, obvious applications of commutativity, distributivity, et cetera, can go without a comment. Nor do I expect you to take a single step in each line. But be careful! Label anything that's not an obvious step (e.g., that is an application of a claim previously proven from the axioms, or is a fiendishly clever & non-obvious application of an axiom), and don't take too many complex steps at once. For example, to demonstrate that $(x-y)(x+y) = x^2 - y^2$, I would at this point *not* like to see you start with the line

$$(x-y)(x+y) = x^2 + xy - xy - y^2$$

— that's *three* distinct applications of distributivity, plus one application of commutativity, in one step! But I'm happy with

$$(x-y)(x+y) = x(x-y) - y(x+y) = x^2 - xy - yx - y^2$$

as a first step — here the applications of distributivity have been clearly separated out, and the application of commutativity has been left for the second step.

The main goal of this homework is: at the end, you should be convinced that *any* property of the reals can be established from first principles — from the axioms — if one wishes to do so. I don't want you taking too many liberties, because I don't want you to slip in any unwarranted assumptions. And the more steps you take at once, the more likely it is that an unwarranted assumption might slip in.

After this homework, I'll be more relaxed about first-principle appeals to the axioms.

Reading for this homework

Start by reading Sections 2.1 through 2.5 of the course notes; this section introduces, in a quite gentle way, the notion of "proofs", and gives illustrations of how to present proofs. Then read Sections 3.1 through 3.5 of the course notes, in which axioms P1 through P12 are introduced. Spivak Chapter 1 covers this material, too; my notes are essentially taken directly from Spivak.

Assignment

1. Begin by re-reading the guidelines for Homework 1, and by reading either Spivak, Chapter 1, or the suggested reading from the course notes for this homework.

2. Using only the axioms P1 through P12, the closure of the set of numbers under addition and multiplication, and the assumption $0 \neq 1$, prove each of the following. Justify *every* line of your proof.

   (a) The *cancellation* property of addition: for all real numbers a, b, c, if $a + b = a + c$ then $b = c$.

   (b) The uniqueness of the multiplicative inverse, in a strong form: if a and x are numbers satisfying $ax = a$ and $a \neq 0$ then $x = 1$. (**comment**: this implies that if x is a multiplicative inverse, that is, if $ax = xa = a$ for all $a \neq 0$, then $x = 1$).

   (c) If $x^2 = y^2$ then either $x = y$ or $x = -y$.

   (d) For all $n \geq 2$, $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1})$. (Here you can use the extra facts, not yet proven, that the order of terms in a sum or product doesn't matter, and that the order of parenthesizing a sum or product doesn't matter. You don't have to be overly precise here. Just do enough to convince yourself

   A that the identity is true, and

   B that you could, if forced, and if given enough time, prove it directly from the axioms for any particular n.

   Then remember the identity — it's very useful.)

3. What is wrong with the following proof that $2 = 1$? (Where of course "2" means "$1 + 1$".)

   Let x be any number, and let $y = x$, so that $x^2 = xy$ and $x^2 - y^2 = xy - y^2$. Factorizing both sides, $(x + y)(x - y) = y(x - y)$ and so $x + y = y$. Using $y = x$ this says $2y = y$ so $2 = 1$.

4. In this question, you must use *only* the axioms P1 through P12, the closure of the set of numbers under addition and multiplication, and the assumption $0 \neq 1$, together with the definitions of $<$ and $>$. I expect you to verify each property *very carefully.* In the world of inequalities, many facts are far from obvious (why does multiplying by a negative change the direction of the inequality, for example?), so I want you to leave

this question absolutely convinced that the order axioms really are all that is needed to create a notion of $<$ and $>$ that agrees with our intuitive understanding.

After you have started proving some of the properties asked for in the question, you can use those to establish later properties, if it seems appropriate.

(a) If $a < b$ and $c < d$ then $a + c < b + d$.

(b) If $a < b$ then $-b < -a$.

(c) If $a < b$ and $c < 0$ then $ac > bc$.

(d) If $0 \leq a < b$ and $0 \leq c < d$ then $ac < bd$.

(e) If $a, b \geq 0$ and $a^2 < b^2$ then $a < b$.

5. Prove that if x and y are not both 0, then

    (a) $x^2 + xy + y^2 > 0$.

    (b) $x^4 + x^3y + x^2y^2 + xy^3 + y^4 > 0$.

6. (a) Show that $(x + y)^2 = x^2 + y^2$ only if[1] $x = 0$ or $y = 0$.

    (b) Show that $(x + y)^3 = x^3 + y^3$ only if $x = 0$ or $y = 0$ or $x = -y$.

    (c) Using the fact that $(x+y)^2$ is not negative, show that $4x^2 + 6xy + 4y^2 > 0$ unless[2] x and y are both 0.

    (d) Use part (c) to find all x, y for which $(x + y)^4 = x^4 + y^4$.

7. This question and the next question are concerned with the *absolute value* function. We may not get to this in class on Wednesday, but that's ok — I'll define it here, and you'll get the chance to think about it before we see it in class... (and/or you can look at Section 3.6 of the course notes).

    The *absolute value* of a real number x, denoted $|x|$ is defined to be

$$x, \text{ if } x \geq 0$$

and

$$-x, \text{ if } x < 0.$$

[1] What does "only if" mean? "p only if q" means that if q doesn't happen, then neither does p, i.e., "(not q) implies (not p)", which is the contrapositive of (and so equivalent to) "p implies q". So in this particular question, you being asked to show that **if** $(x + y)^2 = x^2 + y^2$ **then** either $x = 0$ or $y = 0$; not the other direction (which is somewhat trivial).

Note that this explains the language "if and only if": when we say "p if and only if q", we are saying "p only if q" — i.e., "p implies q" — *and* "p if q" — i.e., "q implies p".

[2] What does "unless" mean? "p unless q" means that the only way for p not to happen, is for q to happen, that is, "(not p) only if q". As we've seen in the previous footnote, this is the same as "(not p) implies q". So in this particular question, you being asked to show that **if** $4x^2 + 6xy + 4y^2 \leq 0$ **then** both $x = 0$ and $y = 0$. The contrapositive of this, which might be easier to think about, is that **if** at least one of x, y are not 0 then $4x^2 + 6xy + 4y^2 > 0$.

So, x is the distance from x to 0 on the number line (always a non-negative number). Let ε be a positive number. Prove that if $|x - x_0| < \varepsilon/2$ and $|y - y_0| < \varepsilon/2$ then both of

$$|(x + y) - (x_0 + y_0)| < \varepsilon$$

and

$$|(x - y) - (x_0 - y_0)| < \varepsilon$$

hold. (Not an idle question; we will need these kinds of manipulations when we come to study limits and continuity.)

8. Prove the *reverse triangle inequality*: for all numbers a, b,

$$||a| - |b|| \leq |a - b|.$$

Also, figure out for which choices of a and b there is equality.

(**Hint**: Break your analysis into cases, choosing the cases in such a way that inside each case, you can remove all the absolute values from both sides of the inequality that you are trying to prove.)

9. In class we used the distributive axiom P9 to show that $a \cdot 0 = 0$ for all real numbers a.

   (a) Show that it is *necessary* to use P9 (or something like it) to prove this. More specifically: find a set of "numbers", that includes special "numbers" "0" and "1" (different from each other), for which there is a notion of "addition" and "multiplication", that satisfies all of the axioms P1 through P8, but for which it is *not* the case that $a \cdot 0 = 0$ for all numbers a.

   (b) P9 must fail in the system you discovered in the previous part (otherwise, you could prove that $a \cdot 0$ always equal 0, using the proof we had in class). Give an explicit example of the failure of the distributive axiom in your system.

Some extra credit problems

Please submit these on a *separate* sheet.

1. (Related to the current material) Find all numbers x, y such that

$$(x+y)^5 = x^5 + y^5.$$

2. (Unrelated) In the picture below, which of the two shaded in regions (the red region and the blue region, if you are looking at the pdf online) has the greater area? (The boundary is a perfect quarter-circle. The two circle-like curves inside the quarter circle are perfect semi-circles, whose diameters are radii of the quarter-circle.)