Math 10850, fall 2019

First midterm exam, Friday October 4

NAME:

Instructions

- The exam goes from 11.30am to 12.30am on Friday, October 4.
- There are 5 questions.
- Present your answers in the space provided. **Use the back of each page if necessary**; if you do, clearly indicate this!
- Present your answers clearly and neatly. Remember that the exam is a chance for you both to show me what you know, and a chance to show me that you can *clearly tell me* what you know. Your proofs should not consist of a collection of unconnected statements, but instead should form a narrative that makes the thread of your logic clear.
- Please read each part **carefully** before answering it; make sure you are answering the question that actually has been asked!
- **Justify all your assertions**, even if a question does not explicitly say this. Partial credit can be given, but only if your answers are supported.
- Calculators are not allowed, nor should they be needed.
- No notes, books or any other external resources are allowed. For your convenience, I've included some of the axioms of real numbers on the last page.
- Remember the Academic Code of Honor Pledge:

  "As a member of the Notre Dame community, I acknowledge that it is my responsibility to learn and abide by principles of intellectual honesty and academic integrity, and therefore I will not participate in or tolerate academic dishonesty."

MAY THE ODDS BE EVER IN YOUR FAVOR!

Question	score	out of
1		10
2		10
3		10
4		10
5		10
Total		50

Using:

a	b	$a \Rightarrow b$
T	T	T
T	F	F
F	T	T
F	F	T

Problems

1. This question concerns the logical statement $p \Rightarrow (q \Rightarrow r)$.

   (a) (4 points) Write down the truth table of $p \Rightarrow (q \Rightarrow r)$. I've given you a template, with space in the middle for auxiliary columns if you want/need them (plus space at the end, that you may want to use later).

p	q	r	$q \Rightarrow r$	$p \Rightarrow (q \Rightarrow r)$	$p \Rightarrow q$	$(p \Rightarrow q) \Rightarrow r$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

   (b) (4 points) Write down an expression involving p, q, r and some (or all) of the logical operators $\wedge$, $\vee$ and $\neg$, that is logically equivalent to $p \Rightarrow (q \Rightarrow r)$.

Using that $a \Rightarrow b$ is equivalent to $\neg a \vee b$, have $p \Rightarrow (q \Rightarrow r)$ is equivalent to $\neg p \vee (q \Rightarrow r)$, which is equivalent to $\neg p \vee (\neg q \vee r)$

   (c) (2 points) Is $p \Rightarrow (q \Rightarrow r)$ equivalent to $(p \Rightarrow q) \Rightarrow r$? *Briefly* justify.

No. See the truth table above; when

p is false
q is true
r is false,

$p \Rightarrow (q \Rightarrow r)$ is true, but
$(p \Rightarrow q) \Rightarrow r$ is false.

(also, they differ when all three are false)

2. (a) (5 points) Using *only* the axioms of the real numbers (see the last page), and the basic properties of equality, show that for all a, $a \cdot 0 = 0$. Each step must follow from an axiom. You cannot use any results we might have proven in class, unless you re-prove them. You should say which axioms you are using in each step (number or name, either is fine). You don't need to mention that you are using basic properties of equality.

Let $a \in \mathbb{R}$ be given.

Have $0+0=0$ (P2),

So $a(0+0)=a0$,

So $a.0+a.0=a.0$ (P9)

So $(a.0+a.0)+(-(a.0)) = a.0+(-(a.0))$, (P3 - existence of $-(a.0)$)

So $a.0+(a.0+(-a.0)) = 0$ (P1 on left, P3 on right)

So $a.0+0=0$ (P3)

So $a.0=0$ (P2).

(b) (5 points) Using *only* the axioms of the real numbers, the basic properties of equality, and (if needed) the result of the last part, show that for all a, b, if both $a \neq 0$ and $b \neq 0$ then $a \cdot b \neq 0$. (You can be more relaxed here about labelling every step. Just make sure the thread of your proof is clear.)

Proof by contradiction: Suppose $a \neq 0$, $b \neq 0$, and $a.b=0$.

Then $(a^{-1})(ab) = (a^{-1})0$,

So $(a^{-1}a)b = 0$ (by part a))

So $1b=0$,

So $b=0$, contradiction.

Proof by contrapositive: Suppose $ab=0$. We want to show that at least one of $a,b=0$.

If $a=0$, we are done.

If $a\neq 0$, then $a^{-1}(ab)=a^{-1}0$,

So $(a^{-1}a)b=0$ (part a))

So $1b=0$,

So $b=0$, and we are done.

3. (a) (4 points) Give a complete and clear statement of the principle of mathematical induction.

Let $P(n)$ be a predicate, with universe of discourse for n being $\mathbb{N}$.

If $(P(1)$ is true and $(\forall n \in \mathbb{N})(P(n)$ implies $P(n+1)))$

then $P(n)$ is true for all $n \in \mathbb{N}$

(b) (6 points) Prove that if a is any real number, and if $b_1, b_2, \ldots, b_n$ are any n real numbers, $n \geq 2$, then

$$a \cdot (b_1 + b_2 + \cdots + b_n) = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_n.$$

You may assume the generalized associativity axiom. Be careful to lay out your proof clearly.

We prove this by induction on $n \geq 2$.

Base Case $n = 2$ is P9

Induction step: Suppose claim is true for some $n \geq 2$.

Let $a, b_1, \ldots, b_n, b_{n+1}$ be real numbers.

Have $a(b_1 + \ldots + b_n + b_{n+1}) = a((b_1 + \ldots + b_n) + b_{n+1})$

$= a(b_1 + \ldots + b_n) + ab_{n+1}$ (P9)

$= ab_1 + \ldots + ab_n + ab_{n+1}$

(by induction hypothesis).

So by induction, claim is true for all $n \in \mathbb{N}$.

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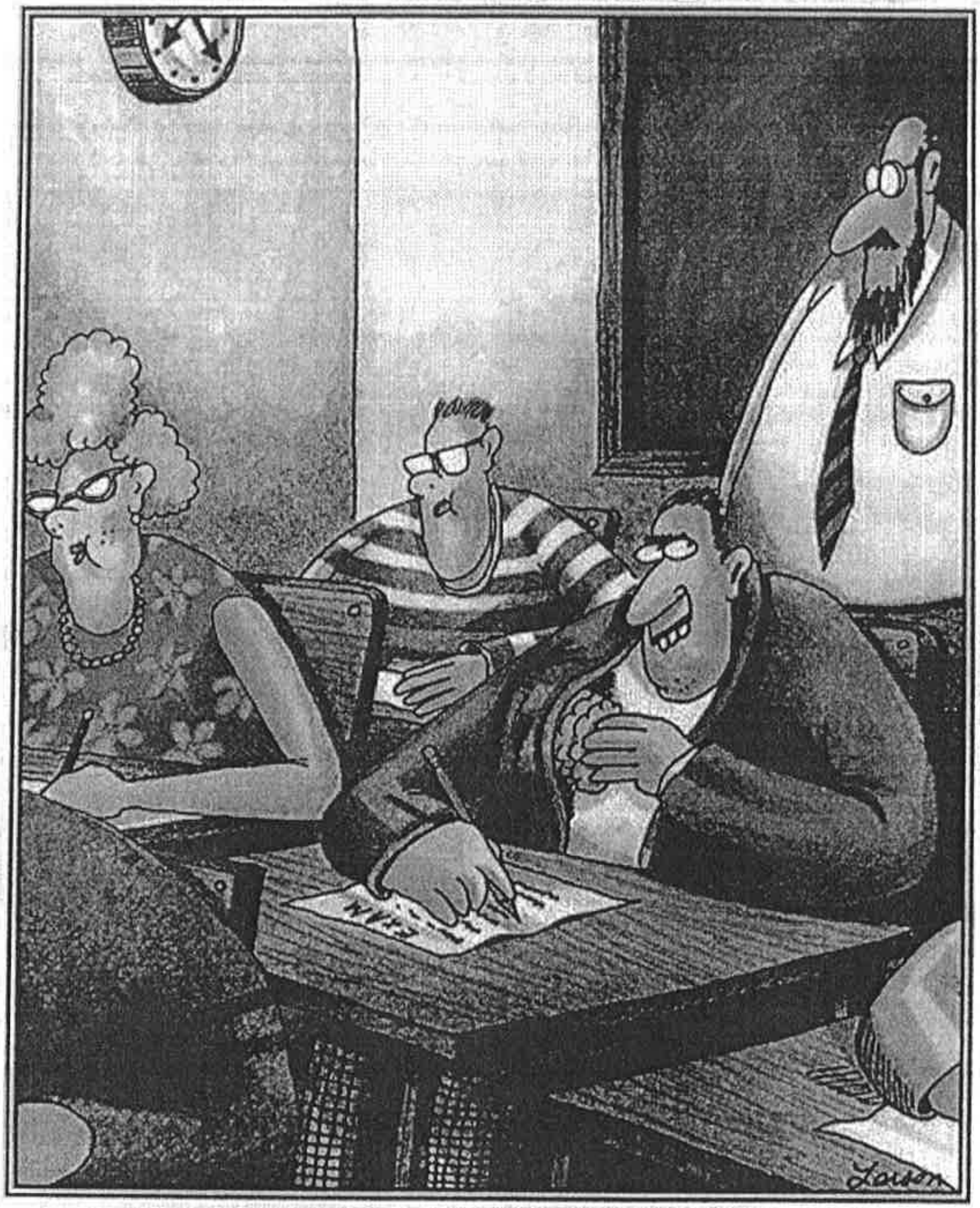

Midway through the exam, Allen pulls out a bigger brain.

The Far Side by Gary Larson

4. (a) (3 points) Give a definition of the binomial coefficient $\binom{n}{k}$ for integers $n \geq k \geq 0$. (There are two acceptable answers).

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

OR $\binom{n}{k}$ = number of subsets of size k of a set of size n.

(b) (4 points) For $n \geq k \geq 1$, $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. Prove this!

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n}{k}\,\frac{(n-1)!}{(k-1)!\,(n-1-(k-1))!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}.$$

$n\binom{n-1}{k-1}$ counts # of ways of choosing a committee of k people from n, with a designated chair, by first selecting the chair, then the rest of the committee.

OR: $k\binom{n}{k}$ counts the same thing, by first selecting the committee and then the chair. Since both count the same thing, they are equal, ie $n\binom{n-1}{k-1} = k\binom{n}{k}$ or $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

(c) (3 points) When $(2x^2 + \frac{1}{x})^9$ is fully expanded out, one of the terms is constant — it doesn't depend on x. Find that constant.

Binomial theorem says

$$\left(2x^2+\frac{1}{x}\right)^9 = \sum_{k=0}^{9} \binom{9}{k} (2x^2)^{9-k} \left(\frac{1}{x}\right)^k$$

$k=6$ is the term that is constant $\left((x^2)^{9-6}\left(\frac{1}{x}\right)^6 = 1 \right)$

So the constant is $\binom{9}{6} 2^{9-6} = \frac{8\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{6\times 5\times 4\times 3\times 2\times 1\times 3\times 2\times 1}$

$= 672$

5. (a) (3 points) Give the (formal) definition of a *function*.

A function is a set of ordered pairs with the property that any element that appears as a first co-ordinate of a pair, appears as the first coordinate of only one pair.

(b) (4 points) Let $f : [-2, 2] \to \mathbb{R}$ be given by $f(x) = |x^2 - 4| + |x^2 - 1| - 4$. Find all x for which $f(x) \geq 0$. Write your answer in interval notation. (Note the domain of f.)

Consider cases. Case 1, $-2 \leq x \leq -1$. Here $f(x) = 4 - x^2 + x^2 - 1 - 4 = -$

So $f(x) \geq 0$ never in this case

Case 2: $-1 \leq x \leq 1$. Here $f(x) = 4 - x^2 + 1 - x^2 - 4$

$= 1 - 2x^2$.

$f(x) \geq 0$ iff $1 - 2x^2 \geq 0$ iff $x^2 \leq \frac{1}{2}$

iff $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$,

all values in range $-1 \leq x \leq 1$.

Case 3: $1 \leq x \leq 2$. As in Case 1, $f(x) \geq 0$ never here

Summary: $f(x) \geq 0$ for $x \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

(c) (3 points) Let h be the function that maps x to $\sqrt{x}$, and let f be the same function from part (b). Find the domains of both $h \circ f$ and $f \circ h$.

- $h \circ f$: $x \in \text{Domain}(h \circ f)$ precisely when $f(x) \geq 0$, (those are the x for which $\sqrt{f(x)}$ makes sense).

  By part (b), $\text{Domain}(h \circ f)$ is $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$

- $f \circ h$: $x \in \text{Domain}(h \circ f)$ if i) $x \geq 0$ (so $\sqrt{x}$ makes sense)

  and ii) $\sqrt{x} \in \text{Domain}(f)$

  Since $\text{Domain}(f) = [-2, 2]$, require $\sqrt{x} \in [-2, 2]$,

  so require $0 \leq x \leq 4$, or $x \in [0, 4]$

Axioms of real numbers: The real numbers $\mathbb{R}$ is a set, containing two special elements 0 and 1, not equal to each other, equipped with operations $+$ (that assigns to each pair (a,b) an element $a+b$) and $\cdot$ (that assigns to each pair (a,b) an element $a \cdot b$), and also containing a subset $\mathbb{P}$, satisfying the following axioms:

P1 (additive associativity) for all a, b, c, $\boxed{a + (b + c) = (a + b) + c}$

P2 (additive identity) for all a, $\boxed{a + 0 = 0 + a = a}$

P3 (additive inverse) for all a there is $-a$ satisfying $\boxed{a + (-a) = (-a) + a = 0}$

P4 (additive commutativity) for all a, b, $\boxed{a + b = b + a}$

P5 (multiplicative associativity) for all a, b, c, $\boxed{a \cdot (b \cdot c) = (a \cdot b) \cdot c}$

P6 (multiplicative identity) for all a, $\boxed{a \cdot 1 = 1 \cdot a = a}$

P7 (multiplicative inverse) for all $a \neq 0$ there is a^{-1} satisfying $\boxed{a \cdot a^{-1} = a^{-1} \cdot a = 1}$

P8 (multiplicative commutativity) for all a, b, $\boxed{a \cdot b = b \cdot a}$

P9 (distributivity) for all a, b, c, $\boxed{a \cdot (b + c) = a \cdot c + b\dot{c}}$

P10 (trichotomy)

P11 (closure of positives under addition)

P12 (closure of positives under multiplication)

P13 (completeness)