Solutions to Exam 2, Math 10560

1. Which of the following expressions gives the partial fraction decomposition of the function
$$f(x) = \frac{3x^2+2x+1}{(x-1)(x^2-1)(x^2+1)}?$$

   Solution: Notice that (x^2-1) is not an irreducible factor. If we write the denominator in terms of irreducible factors we get
$$f(x) = \frac{3x^2+2x+1}{(x-1)^2(x+1)(x^2+1)}$$
   since $(x^2-1) = (x-1)(x+1)$. Thus we see that the final answer should be
$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{Dx+E}{x^2+1}$$

2. Use the Trapezoidal rule with step size $\Delta x = 1$ to approximate the integral $\int_0^4 f(x)dx$ where a table of values for the function $f(x)$ is given below.

x	0	1	2	3	4
$f(x)$	2	1	2	3	5

   Solution: Using the formula for the trapezoidal rule with Δx=1 we see that
$$\begin{aligned}\int_0^4 f(x)dx \approx& \frac{\Delta x}{2}(f(0)+2f(1)+2f(2)+2f(3)+f(4)) = \frac{1}{2}(2+2+4+6+5)\\ =&\frac{19}{2} = 9.5\end{aligned}$$

3. Evaluate the integral $\displaystyle\int_2^\infty xe^{-x}\,dx$.

   Solution: First we find the indefinite integral using integration by parts: Let $u = x$ and $dv = e^{-x}dx$ so that $du = dx$ and $v = -e^{-x}$. So we have that
$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C$$
   Then we see that
$$\int_2^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_2^b xe^{-x}\,dx = \lim_{b\to\infty}\left(-xe^{-x}-e^{-x}\right)\Big|_2^b$$
$$= \lim_{b\to\infty}\Big((-be^{-b}-e^{-b}) - (-2e^{-2}-e^{-2})\Big) = 0 - (-3e^{-2}) = \frac{3}{e^2}$$

4. Compute the integral

$$\int_{-3}^{3} \frac{1}{(x+2)^3}\,dx.$$

Solution: We have to be careful at the point where the function does not exist, namely $x = -2$. So we see that

$$\int_{-3}^{3} \frac{1}{(x+2)^3}\,dx = \int_{-3}^{-2} \frac{1}{(x+2)^3}\,dx + \int_{-2}^{3} \frac{1}{(x+2)^3}\,dx.$$

We work first on the part $\int_{-2}^{3} \frac{1}{(x+2)^3}\,dx$. We will solve this using u-substitution. If we let $u = x+2$ (so $du = dx$), then the bounds change from $x = -2$ to $u = 0$ and $x = 3$ to $u = 5$. Making the substitution we see that

$$\int_{-2}^{3} \frac{1}{(x+2)^3}\,dx = \int_{0}^{5} \frac{1}{u^3}\,du = \lim_{b\to 0}\left(\int_{b}^{5} u^{-3}\,du\right)$$

$$= \lim_{b\to 0}\left(-\frac{u^{-2}}{2}\right)\Bigg|_{b}^{5} = \lim_{b\to 0}\left(-\frac{5^{-2}}{2} + \frac{b^{-2}}{2}\right) = \lim_{b\to 0}\left(-\frac{1}{50} + \frac{1}{2b^2}\right) = \infty$$

So the integral is **divergent**.

5. Compute the integral

$$\int_{0}^{\frac{\pi}{2}} \cos\left(\cos(x)\right)\sin(x)\,dx.$$

Solution: We solve this by u-substitution. Let $u = \cos(x)$ (so $du = -\sin(x)dx$). Then the bounds of integration change from $x = \frac{\pi}{2}$ to $u = 0$ and from $x = 0$ to $u = 1$. Making the substitutions we get

$$\int_{0}^{\frac{\pi}{2}} \cos\left(\cos(x)\right)\sin(x)\,dx = \int_{1}^{0} -\cos(u)\;du$$

$$= -\sin(u)\Big|_{1}^{0} = -\sin(0) - (-\sin(1)) = \sin(1)$$

6. Which of the following is an expression of the area of the surface formed by rotating the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ about the x-axis?

Solution: The formula is given by

$$\int_{a}^{b} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

where in our situation $a = 0$, $b = \frac{\pi}{2}$, $y = \sin(x)$ and so $\frac{dy}{dx} = \cos(x)$. Plugging all in and pulling the 2π out we get:

$$2\pi\int_{0}^{\frac{\pi}{2}} \sin(x)\sqrt{1+\cos^2(x)}\;dx$$

7. Find the centroid of the region bounded by $y = e^x$, $y = 0$, $x = 0$ and $x = 1$.

Solution: First we note that the area of the region A is given by

$$A = \int_0^1 e^x \, dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$$

Now, we find the centroid by finding $\overline{x}$ and $\overline{y}$:

$$\overline{x} = \frac{1}{A} \int_0^1 xe^x \, dx, \quad \overline{y} = \frac{1}{A} \int_0^1 \frac{1}{2}(e^x)^2 \, dx$$

For $\overline{x}$, we solve the integral using integration by parts with $u = x$ and $dv = e^x dx$ so that $du = dx$ and $v = e^x$. Then we get that $\int xe^x \, dx = xe^x - \int e^x dx = xe^x - e^x + C$. Using this we get

$$\overline{x} = \frac{1}{A} \int_0^1 xe^x \, dx = \frac{1}{e-1} \left(xe^x - e^x \right) \Big|_0^1 = \frac{1}{e-1} \left((e - e) - (0 - 1) \right) = \frac{1}{e-1}$$

For $\overline{y}$ we note that $(e^x)^2 = e^{2x}$. Then we use u-substitution with $u = 2x$ so that $du = 2dx$ and the bounds change from $x = 0$ to $u = 0$ and from $x = 1$ to $u = 2$. Making the substitution we get

$$\begin{aligned} \overline{y} =& \frac{1}{A} \int_0^1 \frac{1}{2}(e^x)^2 \, dx = \frac{1}{(e-1)} \frac{1}{4} \int_0^2 e^u \, du = \frac{1}{4(e-1)} \left(e^u \right) \Big|_0^2 \\ =& \frac{1}{4(e-1)} \left(e^2 - 1 \right) = \frac{e+1}{4}. \end{aligned}$$

Thus the centroid lies at the coordinates $\left(\frac{1}{e-1}, \frac{e+1}{4} \right)$.

8. Use Euler's method with step size 0.5 to estimate $y(2)$ where $y(x)$ is the solution to the initial value problem

$$y' = (x - 1)(y - x), \qquad y(1) = 2.$$

Solution: This will require two steps in Euler's method. For step one, we know that $x_0 = 1$ and $y_0 = 2$. Additionally, we know that $h = 0.5$. We also know that $x_1 = 1.5$ and $x_2 = 2$ so we can stop at step 2.

$$\begin{aligned} y_1 &= y_0 + h(x_0 - 1)(y_0 - x_0) = 2 + (.5)(0)(1) = 2 \\ y_2 &= y_1 + h(x_1 - 1)(y_1 - x_1) = 2 + (.5)(1.5 - 1)(2 - 1.5) = 2 + (.5)^3 = 2.125 \end{aligned}$$

9. Compute the arc length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 0$ to $x = 3$.

Solution: We see that $\frac{dy}{dx} = x^{\frac{1}{2}} = \sqrt{x}$. Plugging into the formula for arc length we get that

$$\begin{aligned} \text{arc length} =& \int_0^3 \sqrt{1 + (\sqrt{x})^2} \, dx = \int_0^3 \sqrt{1 + x} \, dx = \frac{2}{3} \left((x+1)^{\frac{3}{2}} \right) \Big|_0^3 \\ =& \frac{2}{3} \left(4^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = \frac{2}{3}(8 - 1) = \frac{14}{3} \end{aligned}$$

10. Compute the integral

$$\int \frac{x^2+2x}{x^2-1}\,dx.$$

Solution: First we do long division dividing x^2-1 into x^2+2x. Doing this we get that

$$\frac{x^2+2x}{x^2-1} = 1 + \frac{2x+1}{x^2-1}$$

and

$$\int \frac{x^2+2x}{x^2-1}\,dx = \int 1\,dx + \int \frac{2x+1}{x^2-1}\,dx \tag{1}$$

The first integral in (1) is straightforward: $\int 1\,dx = x + C$. The second integral is obtained using integration by partial fractions. By partial fractions we obtain:

$$\frac{2x+1}{x^2-1} = \frac{2x+1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

So we have that

$$2x+1 = A(x-1) + B(x+1)$$

Plugging in $x=1$ gives $2B=3$ and plugging in $x=-1$ gives $-2A=-1$, so we see that $A=\frac{1}{2}$ and $B=\frac{3}{2}$. Using this decomposition gives

$$\int \frac{2x+1}{x^2-1}\,dx = \int \frac{\frac{1}{2}}{x+1}\,dx + \int \frac{\frac{3}{2}}{x-1}\,dx = \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C$$

Putting it all together, (1) becomes:

$$\int \frac{x^2+2x}{x^2-1}\,dx = x + \frac{1}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + C$$

11. Evaluate the integral

$$\int_0^1 (1-\sqrt{x})^8\,dx.$$

Solution: We do this with u-substitution. Let $u = 1-\sqrt{x}$ so that $\sqrt{x} = 1-u$ and hence $x = (1-u)^2$. Using this, we see that $dx = -2(1-u)du$. Also, the bounds of integration go from $x=0$ to $u=1$ and from $x=1$ to $u=0$. Making the substitution gives:

$$\int_0^1 (1-\sqrt{x})^8\,dx = \int_1^0 -2(1-u)u^8\,du = 2\int_0^1 (u^8-u^9)\,du$$

$$=2\left(\frac{u^9}{9} - \frac{u^{10}}{10}\right)\Bigg|_0^1 = 2\left(\left(\frac{1}{9}-\frac{1}{10}\right)-0\right) = 2\left(\frac{1}{90}\right) = \frac{1}{45}.$$

12. Find the solution to the initial value problem

$$(1-x)y' - y^2 = 1, \qquad y(2) = 1.$$

Solution: We can make this into a separable equation in the following way:

$$(1-x)y' = y^2 + 1$$

Now, separate and integrate to find the solution:

$$\frac{1}{y^2+1}dy = \frac{1}{1-x}dx$$

and so

$$\int \frac{1}{y^2+1}dy = \int \frac{1}{1-x}dx$$
$$\tan^{-1}(y) = -\ln|x-1| + C$$

To solve for C we use the initial value $y(2) = 1$ giving us that $\tan^{-1}(1) = -\ln|2-1| + C$ which implies that $C = \tan^{-1}(1) = \frac{\pi}{4}$. Solving for y we get

$$y = \tan\left(\frac{\pi}{4} - \ln(x-1)\right)$$

13. Solve the initial value problem

$$y' = \frac{2x-y}{1+x}, \qquad y(1) = 2.$$

Solution: We first rewrite it as $y' = \frac{2x}{1+x} - \frac{y}{1+x}$ which allows us to rewrite as

$$y' + \frac{y}{x+1} = \frac{2x}{x+1}$$

Now, it is in standard form for a first-order *linear* differential equation with $P(x) = \frac{1}{x+1}$ and $Q(x) = \frac{2x}{x+1}$. We find the integrating factor (noting $\int P(x)dx = \int \frac{1}{x+1}\,dx = \ln|x+1|$):

$$I(x) = e^{\int P(x)dx} = e^{(\ln|x+1|)} = x+1.$$

So the final solution is given by

$$\begin{aligned} y(x) =& \frac{1}{I(x)}\left(\int I(x)Q(x)\,dx\right) = \frac{1}{x+1}\left(\int (x+1)\left(\frac{2x}{x+1}\right)\,dx\right) \\ =& \frac{1}{x+1}\int 2x\,dx = \frac{1}{x+1}\left(x^2 + C\right) \end{aligned}$$

Using the initial value $y(1) = 2$ tells us that $2 = \frac{1}{2}(1+C)$ which means $C = 3$. So finally we have that

$$y(x) = \frac{x+1}{x^2+3}$$