SOLUTIONS TO EXAM 1, MATH 10560

1. *The function $f(x) = \ln x - \frac{1}{x}$ is one-to-one. Compute $(f^{-1})'(-1)$.*

Solution: We have

$$(f^{-1})'(-1) = \frac{1}{f'(f^{-1}(-1))} \ ,$$

$f^{-1}(-1) = 1$ and $f'(x) = \frac{1}{x} + \frac{1}{x^2}$. Hence $f'(f^{-1}(-1)) = f'(1) = 2$, and $(f^{-1})'(-1) = \frac{1}{2}$.

2. *Differentiate the function*

$$f(x) = (2x)^x \ .$$

Solution: Use logarithmic differentiation:

$$\ln f = x \ln(2x),$$

$$\frac{f'}{f} = \ln(2x) + x \cdot \frac{2}{2x} = \ln(2x) + 1,$$

$$f'(x) = (2x)^x \left(\ln(2x) + 1\right).$$

3. *Compute the integral*

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx \ .$$

Solution: Make the substitution $u = e^x$ with $du = (e^x + 1)dx$; when $0 < x < \ln 2$, we have $2 < u < 3$. Thus

$$\int_0^{\ln 2} \frac{e^x}{1+e^x} dx = \int_2^2 \frac{3}{1+u} du = [\ln |u|]_2^3 = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right) \ .$$

4. *Simplify the expression*

$$\log_2 \left(\frac{2^{x^2+1}}{4^x}\right) \ .$$

Solution:

$$\log_2 \left(\frac{2^{x^2+1}}{4^x}\right) = \log_2(2^{x^2+1}) - \log_2(4^x) = (x^2+1) - \log_2(2^{2x}) = x^2 + 1 - 2x = (x-1)^2 \ .$$

5. *A savings account has a yearly interest rate of r. Let $y(t)$ be the balance of the savings account after t years, and suppose the compounding of interest on the account is such that $y(t)$ satisfies the condition $y'(t) = ry(t)$. For which value of r will your initial investment triple in 15 years?*

Solution: Any solution of the differential equation $y'(t) = ry(t)$ is of the form $y(t) = y_0 e^{rt}$, where y_0 is the initial investment. So $3y_0 = y_0 e^{15r}$, and $3 = e^{15r}$. Taking the logarithm we get $\ln 3 = 15r$, so $r = \frac{1}{15} \ln 3$.

6. *Compute* $\tan^{-1}\left(\tan\frac{7\pi}{5}\right)$.

Solution: Note that the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, so $\frac{7\pi}{5}$ is not the answer. Now for any α one has $\tan(\pi+\alpha)=\tan(\alpha)$. So $\tan\frac{7\pi}{5}=\tan\left(\pi+\frac{2\pi}{5}\right)=\tan\frac{2\pi}{5}$. And finally, $\tan^{-1}\left(\tan\frac{7\pi}{5}\right)=\tan^{-1}\left(\tan\frac{2\pi}{5}\right)=\frac{2\pi}{5}$, since $-\frac{\pi}{2}<\frac{2\pi}{5}<\frac{\pi}{2}$.

7. *Simplify* $\sec(\tan^{-1}x)$.

Solution: Let $y=\tan^{-1}x$. Then $\tan y=x$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$. Note that $\sec^2 y=1+\tan^2 y$. Since $\sec y>0$ for $-\frac{\pi}{2}<y<\frac{\pi}{2}$, we have $\sec y=\sqrt{1+\tan^2 y}$. Thus $\sec(\tan^{-1}x)=\sqrt{1+\tan^2(\tan^{-1}x)}=\sqrt{1+x^2}$.

8. *Find the limit*

$$\lim_{x\to 0}\frac{\sinh(x)-x}{x^3}\ .$$

Solution: We have an indeterminate form $\frac{0}{0}$, hence apply l'Hospital's Rule:

$$\begin{aligned}\lim_{x\to 0}\frac{\sinh(x)-x}{x^3}&=\lim_{x\to 0}\frac{\cosh(x)-1}{3x^2}\quad\text{(l'Hospital's Rule)}\\&=\lim_{x\to 0}\frac{\sinh(x)}{6x}\quad\text{(l'Hospital's Rule)}\\&=\lim_{x\to 0}\frac{\cosh(x)}{6}=\frac{1}{6}\ .\end{aligned}$$

9. *Evaluate the integral*

$$\int_0^{\pi/2}\sin^3(x)\cos^3(x)dx.$$

Solution: Use the identity $1-\cos^2(x)=\sin^2(x)$:

$$\begin{aligned}\int_0^{\pi/2}\sin^3(x)\cos^3(x)dx&=\int_0^{\pi/2}(1-\cos^2(x))\sin(x)\cos^3(x)dx\\&=-\int_1^0(u^3-u^5)du\quad(u=\cos(x),\ du=-\sin(x)dx)\\&=\int_0^1(u^3-u^5)du=\left[\frac{u^4}{4}-\frac{u^6}{6}\right]_0^1=\frac{1}{4}-\frac{1}{6}=\frac{1}{12}.\end{aligned}$$

Note that you can also solve by doing a $u=\sin(x)$ substitution using the identity $1-\sin^2(x)=\cos^2(x)$.

10. *Evaluate the limit*

$$\lim_{x\to 0}\left(\cosh(x)\right)^{1/x^2}.$$

Solution: The limit has indeterminate form 1^∞. Let $L = \lim_{x\to 0} \big(\cosh(x)\big)^{1/x^2}$.

$$\begin{aligned}\ln L &= \lim_{x\to 0} \ln\left(\big(\cosh(x)\big)^{1/x^2}\right)\\ &= \lim_{x\to 0} \frac{\ln\big(\cosh(x)\big)}{x^2}\\ &= \lim_{x\to 0} \frac{\tanh(x)}{2x} \quad \text{(l'Hospital's rule)}\\ &= \lim_{x\to 0} \frac{\operatorname{sech}^2(x)}{2} \quad \text{(l'Hospital's rule)}\\ &= \frac{1}{2}.\end{aligned}$$

Therefore $L = e^{\frac{1}{2}}$.

11. *Compute the integral*

$$\int_0^1 4\tan^{-1}(x)dx \ .$$

Solution:

$$\begin{aligned}&\int_0^1 4\tan^{-1}(x)dx \quad \text{(integration by parts with } u = \tan^{-1} x, dv = dx)\\ &= \left[4x\tan^{-1}(x)\right]_0^1 - 4\int_0^1 \frac{x}{1+x^2}dx\\ &\text{(substitution } u = 1+x^2,\ du = 2xdx)\\ &= \pi - 2\left[\ln(1+x^2)\right]_0^1 = \pi - 2\ln 2 = \pi - \ln 4.\end{aligned}$$

12. *Evaluate the integral*

$$\int 2x(\ln x)^2 dx \ .$$

Solution:

$$\begin{aligned}&\int 2x(\ln x)^2 dx \quad \text{(integration by parts with } u = \ln^2 x,\ dv = 2xdx)\\ &= x^2\ln^2 x - \int 2x\ln x dx \quad \text{(integration by parts with } u = \ln x,\ dv = 2xdx)\\ &= x^2\ln^2 x - \left(x^2\ln x - \int \frac{x^2}{x}dx\right) = x^2\ln^2 x - x^2\ln x + \frac{1}{2}x^2 + C.\end{aligned}$$

13. *Calculate the integral*

$$\int \sqrt{4-x^2}\ dx \ .$$

Solution: Use trigonometric substitution $x = 2\sin\theta$; then $\theta = \sin^{-1}\left(\frac{x}{2}\right)$ and $dx = 2\cos\theta d\theta$. Hence

$$
\begin{aligned}
\int \sqrt{4-x^2}\;dx &= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta \\
&= \int 4\cos\theta\cdot\cos\theta\,d\theta = 2\int 2\cos^2\theta\,d\theta \quad (\text{use } 2\cos^2\theta = 1+\cos 2\theta) \\
&= 2\int (1+\cos 2\theta)d\theta = 2\left(\theta + \frac{1}{2}\sin 2\theta\right) \quad (\text{use } \sin 2\theta = 2\sin\theta\cos\theta) \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\sin\left(\sin^{-1}\left(\frac{x}{2}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C.
\end{aligned}
$$