Name: ______________________________

Instructor: __________________________

Math 10550, Exam 1, Solutions
September 20, 2011.

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for 1 hour and 15 min.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 13 pages of the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
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8.	(a)	(b)	(c)	(d)	(e)
9.	(a)	(b)	(c)	(d)	(e)
10.	(a)	(b)	(c)	(d)	(e)

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Multiple Choice	________
11.	________
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Multiple Choice

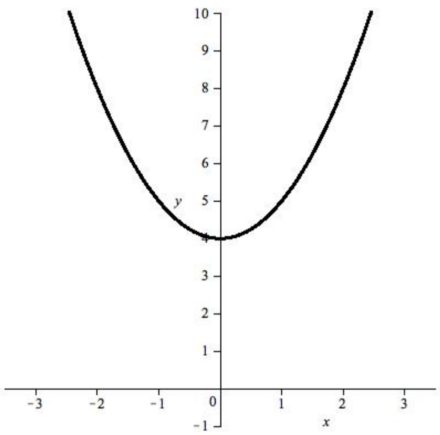
1.(6 pts.) Let $f(x) = x^2$ and $g(x) = x + 3$. Which of the following is the graph of the equation

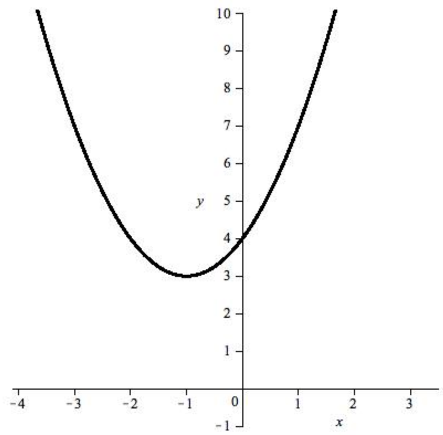
$$y = 1 + f(g(x))?$$

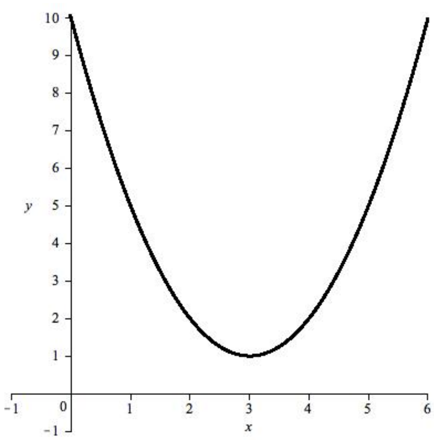
(Note that the label for each graph is given on the lower left of the graph.)

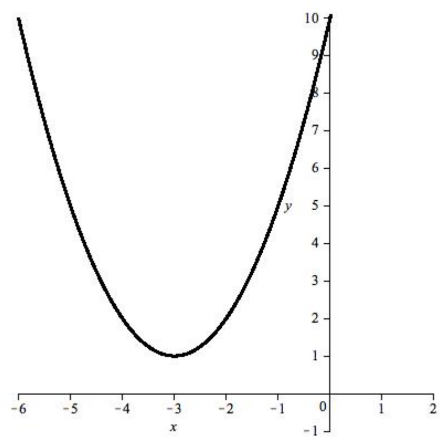
$$y = 1 + f(g(x)) = 1 + f(x+3) = 1 + (x+3)^2.$$

The graph of this curve is found by shifting the graph of $y = x^2$ to the left by three units and upwards by one unit. (d) below.

(a) 

(b) 

(c) 

(d) 

(e) None of the above

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2.(6 pts.) For what value of c is the function f given by

$$f(x) = \begin{cases} \dfrac{c\sqrt{x} - c}{x - 1} & x \geq 1 \\ \\ x - c & x < 1 \end{cases}$$

continuous everywhere? This function is continuous on the interval $(-\infty, 1)$ since the graph of $y = x - c$ is continuous on that interval. Similarly the function f is continuous on the interval $(1, \infty)$ since the graph of $y = \dfrac{c\sqrt{x} - c}{x - 1}$ is continuous on that interval for any value of c.

This function is continuous at $x = 1$ if $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$.

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - c) = 1 - c.$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} \frac{c\sqrt{x} - c}{x - 1} = \lim_{x \to 1^-} \frac{c(\sqrt{x} - 1)}{x - 1} = \lim_{x \to 1^-} \frac{c(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)}$$

$$= \lim_{x \to 1^-} \frac{c(x - 1)}{(x - 1)(\sqrt{x} + 1)} = \lim_{x \to 1^-} \frac{c}{(\sqrt{x} + 1)} = \frac{c}{2}.$$

Now f is continuous at $x = 1$ if $1 - c = \dfrac{c}{2}$ or $c = 2 - 2c$, that is $3c = 2$ or $c = \dfrac{2}{3}$.

(a) $c = 1$ (b) $c = \dfrac{2}{3}$ (c) $c = \dfrac{1}{2}$

(d) $c = 0$ (e) $c = -\dfrac{1}{2}$

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3.(6 pts.) Compute $\lim_{x\to 2^-} \dfrac{4-x^2}{x^2-4x+4}$

$$\lim_{x\to 2^-} \frac{4-x^2}{x^2-4x+4} = \lim_{x\to 2^-} \frac{(2-x)(2+x)}{(x-2)(x-2)} = \lim_{x\to 2^-} \frac{-(x-2)(2+x)}{(x-2)(x-2)} = \lim_{x\to 2^-} \frac{-(2+x)}{(x-2)}$$

As x approaches 2 from the left, $-(x+2)$ approaches -4 and is negative.

As x approaches 2 from the left, $(x-2)$ approaches 0 and is negative.

Therefore the limit $\lim_{x\to 2^-} \dfrac{-(2+x)}{(x-2)} = +\infty$.

(a) 2

(b) $-\infty$

(c) $+\infty$

(d) 4

(e) Does not exist and is not ∞ or $-\infty$.

4.(6 pts.) Compute

$$\lim_{x\to \frac{\pi}{2}} \frac{\sin x}{\left(x-\frac{\pi}{2}\right)^2}$$

As x approaches $\frac{\pi}{2}$ from the left $\sin x$ approaches 1 and is positive.

As x approaches $\frac{\pi}{2}$ from the left $\left(x-\frac{\pi}{2}\right)^2$ approaches 0 and is positive.

Therefore the limit $\lim_{x\to \frac{\pi}{2}} \dfrac{\sin x}{\left(x-\frac{\pi}{2}\right)^2} = \infty$.

(a) $+\infty$

(b) $-\infty$

(c) Does not exist and is not ∞ or $-\infty$.

(d) 0

(e) 1

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5.(6 pts.) A particle is moving on a vertical axis. The height of the particle after t seconds is given by the function

$$H(t) = 400 - t^2 - \sqrt{t} \text{ meters.}$$

Which of the following limits gives the velocity of the particle after 4 seconds (when $t = 4$)?

The velocity of the particle at time t is given by $\lim_{h\to 0} \dfrac{H(t+h) - H(t)}{h}$. When $t = 4$, we get that the velocity is equal to

$$\lim_{h\to 0} \frac{H(4+h) - H(4)}{h} = \lim_{h\to 0} \frac{400 - (t+h)^2 - \sqrt{t+h} - [400 - 16 - 2]}{h}$$

$$= \lim_{h\to 0} \frac{400 - (t+h)^2 - \sqrt{t+h} - 382}{h}$$

(a) $\displaystyle \lim_{h\to 4} \frac{400 - (4+h)^2 - \sqrt{4+h}}{h}$

(b) $\displaystyle \lim_{h\to 0} \frac{400 - (4+h)^2 - \sqrt{4+h} - 382}{h}$

(c) $\displaystyle \lim_{h\to 0} \frac{400 - (h)^2 - \sqrt{h} - 382}{h}$

(d) $\displaystyle \lim_{h\to 4} \frac{400 - (4+h)^2 - \sqrt{4+h} - 382}{h}$

(e) $\displaystyle \lim_{h\to 0} \frac{400 - (4+h)^2 - \sqrt{4+h}}{h}$

6.(6 pts.) Let $f(x) = \sqrt[7]{x^3} + \sqrt{x}\sin x$. What is $f'(x)$?

$$f(x) = x^{3/7} + x^{1/2}\sin x$$

Using the power rule and the product rule, we get

$$f'(x) = \frac{3}{7}x^{-(4/7)} + (\sin x)\frac{1}{2\sqrt{x}} + x^{1/2}\cos x.$$

$$= \frac{3}{7\sqrt[7]{x^4}} + \frac{\sin x}{2\sqrt{x}} + \sqrt{x}\cos x.$$

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(a) $\dfrac{3}{7\sqrt[7]{x^4}} + \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

(b) $\sqrt[7]{3x^2} + \dfrac{\sin x}{2\sqrt{x}}$

(c) $\dfrac{3}{7\sqrt[7]{x^4}} + \dfrac{\cos x}{2\sqrt{x}}$

(d) $\sqrt[7]{3x^2} + \sin x + \sqrt{x}\cos x$

(e) $\dfrac{3}{7\sqrt[7]{x^4}} + \dfrac{\sin x}{2\sqrt{x}}$

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7.(6 pts.) Find the equation of the tangent line to $y = x^2 \cos x + 1$ at $x = \frac{\pi}{2}$.

Using the product rule and the summation rule, we get

$$y' = 2x \cos x + x^2(-\sin x).$$

When $x = \frac{\pi}{2}$,

$$y' = 2\frac{\pi}{2} \cos \frac{\pi}{2} - \frac{\pi^2}{4}(\sin \frac{\pi}{2}) = 2\frac{\pi}{2} \cdot 0 - \frac{\pi^2}{4} \cdot 1 = \frac{\pi^2}{4}.$$

Therefore the slope of the tangent to the curve at $x = \frac{\pi}{2}$ is $m = -\frac{\pi^2}{4}$ and a point on the tangent is given by $(\frac{\pi}{2}, \left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 1) = (\frac{\pi}{2}, \left(\frac{\pi}{2}\right)^2 \cdot 0 + 1) = (\frac{\pi}{2}, 1)$.

Therefore, the equation of the tangent to the curve when $x = \frac{\pi}{2}$ is given by

$$y - 1 = -\frac{\pi^2}{4}(x - \frac{\pi}{2}).$$

(a) $y - 1 = (-\frac{\pi^2}{4} + 1)(x - \frac{\pi}{2})$

(b) $y - 1 = -\pi(x - \frac{\pi}{2})$

(c) $y = -\frac{\pi^2}{4}x$

(d) $y - 1 = -\frac{\pi^2}{4}(x - \frac{\pi}{2})$

(e) $y = \pi x + 1$

8.(6 pts.) Let $f(x) = \cos(x^2 + 2x - 1)$. Find $f'(x)$.

$f(x) = g(h(x))$, where $g(x) = \cos x$ and $h(x) = x^2 + 2x - 1$.

By the chain rule,

$f'(x) = g'(h(x))h'(x) = -[\sin(x^2 + 2x - 1)] \cdot (2x + 2) = -(2x + 2)\sin(x^2 + 2x - 1)$.

(a) $(2x + 2)\cos(x^2 + 2x - 1)$

(b) $-\sin(x^2 + 2x - 1)$

(c) $-\sin(x^2 + 2x - 1) + \cos(2x + 2)$

(d) $(2x + 2)\sin(x^2 + 2x - 1)$

(e) $-(2x + 2)\sin(x^2 + 2x - 1)$

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9.(6 pts.) For $f(x) = (x^3 + 2)\sin x$, find $f''(x)$.

Using the product rule, we get $f'(x) = 3x^2 \sin x + (x^3 + 2)\cos x$.

Using the product rule for both terms above, we get

$f''(x) = 6x \sin x + 3x^2 \cos x + 3x^2 \cos x - (x^3 + 2)\sin x = 6x \sin x + 6x^2 \cos x - (x^3 + 2)\sin x$.

(a) $6x \sin x + 6x^2 \cos x - (x^3 + 2)\sin x$

(b) $6x \sin x - (x^3 + 2)\sin x$

(c) $6x \sin x + 3x^2 \cos x - (x^3 + 2)\sin x$

(d) $-6x \sin x$

(e) $6x \sin x - 6x^2 \cos x - (x^3 + 2)\sin x$

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10.(6 pts.) If $f(x) = \dfrac{x^3+2}{x^{100}-x}$, find $f'(x)$.

Using the quotient rule, we get

$$f'(x) = \frac{(x^{100}-x)(3x^2) - (x^3+2)(100x^{99}-1)}{(x^{100}-x)^2}.$$

(a) $\dfrac{(x^{100}-x)(3x^2) + (x^3+2)(100x^{99}-1)}{(x^{100}-x)^2}$

(b) $\dfrac{(x^{100}-x)(3x^2) - (x^3+2)(100x^{99}-1)}{(x^{100}-x)^2}$

(c) $\dfrac{(x^3+2)(100x^{99}-1) - (x^{100}-x)(3x^2)}{(x^{100}-x)^2}$

(d) $\dfrac{3x^2}{100x^{99}-1}$

(e) $\dfrac{(x^{100}-x)(3x^2) - (x^3+2)(100x^{99}-1)}{(x^3+2)^2}$

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Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(10 pts.) Show that the function

$$f(x) = 3x - 1 - x^3$$

has a root in the interval $[1, 2]$.

Make sure to identify which theorem you use and verify that all of the conditions for its use are satisfied for full credit.

f is a continuous function, since it is a polynomial.

We have $f(1) = 1 > 0$ and $f(2) = -3 < 0$.

Therefore by the Intermediate value theorem there is some number c with $1 < c < 2$ for which $f(c) = 0$, giving us a root of the function in the interval $[1, 2]$.

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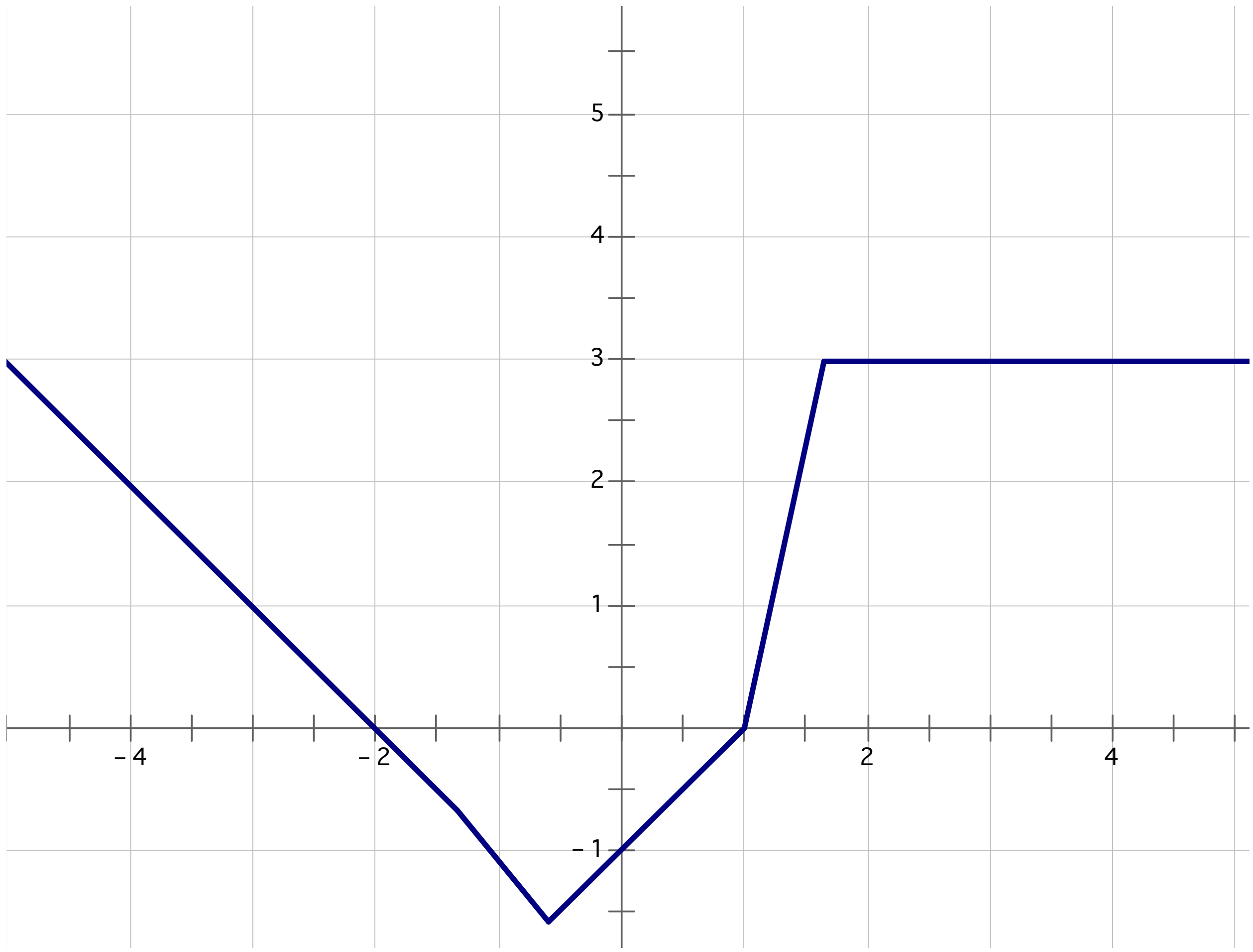
12.(6 pts.) Give a rough sketch of the graph of a continuous function $y = f(x)$ below, for which

$$f(0) = -1 \quad f'(0) = 1, \quad f(2) = 3, \quad f'(2) = 0, \quad f(-2) = 0, \quad f'(-2) = -1,$$



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13.(12 pts.) Consider the curve given by $y = \frac{x^3}{3} + x^2 + x + 1$.

(a) One of the tangent lines to the curve is horizontal. Find its equation.

$$y' = x^2 + 2x + 1.$$

When the tangent line to the curve is horizontal, it has slope $m = 0$.

Therefore the derivative of the function is 0 at the point of tangency.

$$x^2 + 2x + 1 = 0 \text{ if } (x+1)(x+1) = 0 \text{ if } x = -1.$$

Therefore the point of tangency is given by $(-1, \frac{(-1)^3}{3} + (-1)^2 + (-1) + 1) = (-1, \frac{2}{3})$.

The equation of the (horizontal) tangent line is given by

$$y - \frac{2}{3} = 0(x+1) \text{ or } \boxed{y = \frac{2}{3}}.$$

(b) Two of the tangent lines to the curve are parallel to the line $y = x$. Find their equations.

A line parallel to the line $y = x$ has the same slope, $m = 1$.

Since $y' = x^2 + 2x + 1$, a tangent line to the curve has slope $m = 1$ if $x^2 + 2x + 1 = 1$ or $x^2 + 2x = 0$ or $x(x+2) = 0$, that is $\boxed{\text{x = 0} \;\; or \;\; \text{x = -2}}$.

When $x = 0$, the corresponding point on the curve is

$$(0, \frac{(0)^3}{3} + (0)^2 + (0) + 1) = (0, 1)$$

and the tangent line at this point is given by

$$y - 1 = 1(x - 0) \text{ or } \boxed{y = x + 1}.$$

When $x = -2$, the corresponding point on the curve is

$$(-2, \frac{(-2)^3}{3} + (-2)^2 + (-2) + 1) = (-2, \frac{1}{3})$$

and the tangent line at this point is given by

$$y - \frac{1}{3} = 1(x + 2) \text{ or } \boxed{y = x + \frac{7}{3}}.$$

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14.(12 pts.) Consider the following table of function values:

	$x=2$	$x=3$
$f(x)$	2	-1
$g(x)$	$\sqrt{3}$	1
$f'(x)$	$\sqrt{2}$	2
$g'(x)$	$1/2$	$1/2$

(a) Find $(f+g)'(2)$

$$(f+g)'(2) = f'(2) + g'(2) = \sqrt{2} + \frac{1}{2}.$$

(b) Find $\left(\frac{f}{g}\right)'(3)$.

$$\begin{aligned}\left(\frac{f}{g}\right)'(3) &= \frac{g(3)f'(3) - f(3)g'(3)}{(g(3))^2} \\ &= \frac{1\cdot 2 - (-1)\cdot\frac{1}{2}}{1} = 2.5.\end{aligned}$$

(c) Find $h'(2)$ where $h(x) = f([g(x)]^2)$.

$$h'(x) = f'([g(x)]^2)2[g(x)]g'(x).$$

$$\begin{aligned}h'(2) &= f'([g(2)]^2)2[g(2)]g'(2) \\ &= f'([\sqrt{3}]^2)2[\sqrt{3}]\frac{1}{2} \\ &= f'(3)\sqrt{3} \\ &= 2\cdot\sqrt{3}\end{aligned}$$

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