Linear programming, graphically

We've seen examples of problems that lead to linear *constraints* on some unknown quantities.

Now we are going to add an extra ingredient: some quantity that we want to maximize or minimize, such as profit, or costs.

If the quantity to be maximized/minimized can be written as a linear combination of the variables, it is called a linear *objective function*.

Linear programming is the business of finding a point in the feasible set for the constraints, which gives an optimum value (maximum or a minimum) for the objective function.

We'll see how a linear programming problem can be solved graphically.

Example — constraints

A juice stand sells two types of fresh juice in 12 oz cups, the Refresher and the Super-Duper. The Refresher is made from 3 oranges, 2 apples and a slice of ginger. The Super Duper is made from one slice of watermelon, 3 apples and one orange. The owners of the juice stand have 50 oranges, 40 apples, 10 slices of watermelon and 15 slices of ginger. Let x denote the number of Refreshers they make and let y denote the number of Super-Dupers they make.

Last time, we saw that the set of constraints on x and y was:

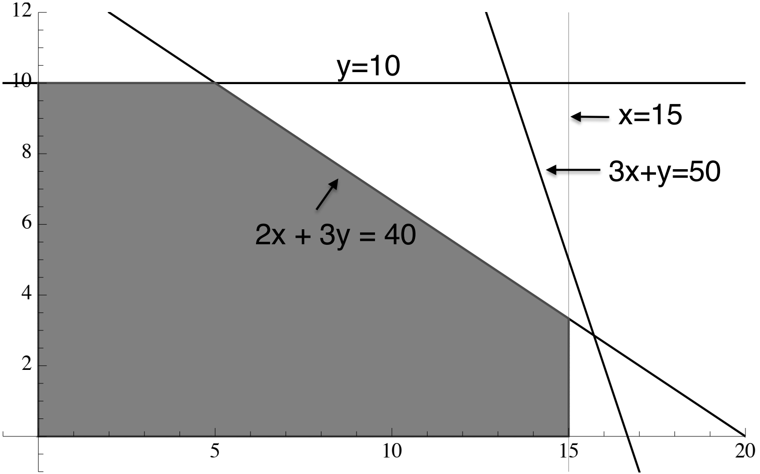
$$3x + y \leqslant 50 \qquad 2x + 3y \leqslant 40$$

$$x \leqslant 15 \qquad y \leqslant 10$$

$$x \geqslant 0 \qquad y \geqslant 0$$

Example — the feasible set

Here is the *feasible set*, the set of combinations of x and y that are possible given the limited supply of ingredients:



Example — adding an objective

Now suppose that Refreshers sell for \$6 each and Super-Dupers sell for \$8 each. Let's suppose also that the juice stand will sell all of the drinks they can make on this day, so their revenue for the day is $6x + 8y$. If a goal of the juice stand is to maximize revenue, then they want to *maximize* the value of $6x + 8y$, given the constraints on production.

In other words they want to find a point (x, y) in the feasible set which gives a maximum value for the **objective function** $6x + 8y$. [Note that the value of the objective function ($6x + 8y =$ revenue) varies as (x, y) varies over the points in the feasible set. For example if $(x, y) = (2, 5)$, revenue $= 6(2) + 8(5) = \$52$, whereas if $(x, y) = (5, 10)$, revenue $= 6(5) + 8(10) = \$110$.]

Terminology

Suppose we are given a problem that involves assigning values x, y to some quantities.

The choices of x, y may be subject to some *constraints*: linear inequalities of the form

$$a_0x + a_1y \leqslant b, \qquad a_0x + a_1y < b,$$

$$a_0x + a_1y \geqslant b, \qquad a_0x + a_1y > b,$$

where a_0, a_1 and b are constants.

There is a linear *objective function*: an expression of the form $cx + dy$, where c and d are constants, and we wish to find the maximum or minimum value that the objective function can take on the feasible set. We use the term *optimal value* to cover both maximizing and minimizing.

A *linear programming* problem is the problem of finding a point $(x_0, y_0) \in F$, the feasible set where all constraints are satisfied, with $O(x_0, y_0)$ as big as possible (if we are doing a maximum problem), or as small as possible (if we are minimizing).

A BIG IDEA of linear programming

If the feasible set of a linear programming problem with two variables is bounded (contained inside some big circle; equivalently, there is no direction in which you can travel indefinitely while staying in the feasible set), then, whether the problem is a minimization or a maximization, there will be an optimum value. Furthermore:

- there will be some *corner point* of the feasible region that is an optimum
- if there is more than one optimum corner point then there will be exactly two of them, they will be adjacent, and any point in the line between them will also be optimum.

The picture on the next page, taken from page 238 of the text, illustrates this graphically.

A BIG IDEA of linear programming

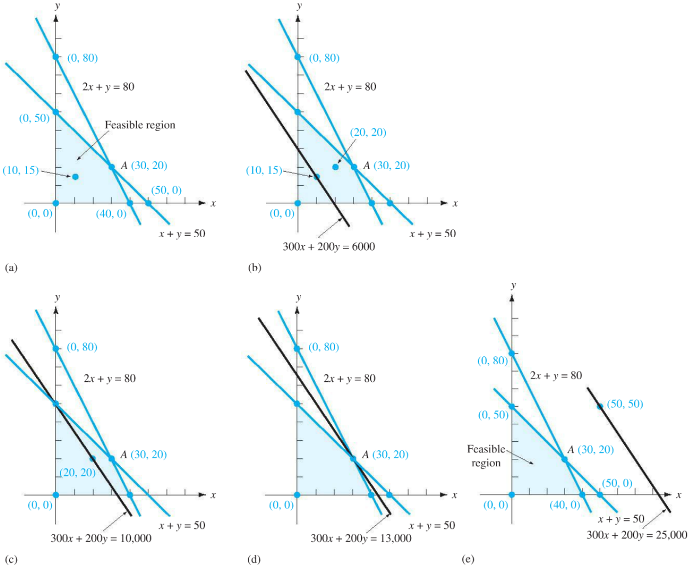
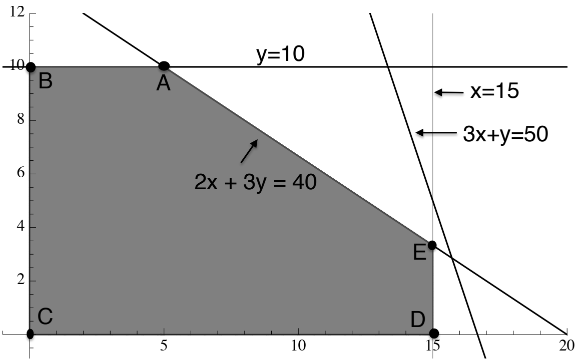


FIGURE 3–16 As an objective function increases, the objective function line moves toward a corner. In (e) the objective function moves out of the feasible region and out of consideration.

Example — solving the problem

We now know that the maximum of $6x + 8y$ on the feasible set occurs at a corner of the feasible set since the feasible set is bounded (it may occur at more than one corner, but it occurs at at least one). We already have a picture of the feasible set and below, we have labelled the corners, A, B, C, D and E.



Example — solving the problem

To find the maximum value of $6x + 8y$ and a point (x, y) in the feasible set at which it is achieved, we need only calculate the co-ordinates of the points A, B, C, D and E and compare the value of $6x + 8y$ at each.

Point	Coordinates	Value of $6x + 8y$
A		
B		
C	(0, 0)	0
D		
E		

Example — solving the problem

Point	Coordinates	Value of $6x + 8y$
A	(5,10)	110
B	(0,10)	80
C	(0, 0)	0
D	(15,0)	90
E	(15,10/3)	$\frac{350}{3}$

Hence E is the largest value (and C is the smallest).

The way to generate optimal revenue is to produce 15 Refreshers and 3.33... Super-Dupers, for a total revenue of \$116.66

Example — solving the problem

BUT: this solution is **not** feasible, since we can only produce whole-number quantities of drinks. We might round down y (to make sure we stay in the feasible region), and report an answer of 15 Refreshers and 3 Super-Dupers, for a total revenue of \$114. If we explore a little more, we find that $x = 14$ and $y = 4$ is also feasible, and yields revenue \$116. This is the optimum.

In general, dealing with integral constraints is **very** tough. In practice, if there are integral constraints, the best way to proceed is to solve the problem without worrying about this additional condition and then look at the nearby integer points to find the maximum.

Example with many solutions

Suppose that Super-Dupers sell for \$9 instead of \$8. What is the new maximum revenue?

Now our revenue function is $6x + 9y$. The feasible set remains unchanged, so the corners remain unchanged. We test the new objective at each corner, as before:

Point	Coordinates	Value of $6x + 9y$
A	(5,10)	120
B	(0,10)	90
C	(0, 0)	0
D	(15,0)	90
E	(15,10/3)	120

Now there are two corners — A and E — that have maximum revenue, \$120. Both of these are solutions, as is any point that lies in a straight line between them.

There is a whole line of solutions because the new revenue line $6x + 9y$ is parallel to the constraint line $2x + 3y = 40$, that joins A and E.

If the feasible set is not bounded

If the feasible set of a linear programming problem is not bounded (there is a direction in which you can travel indefinitely while staying in the feasible set) then a particular objective may or may not have an optimum:

- if it is a maximization problem, there might be a maximum, or it might be possible to make the objective arbitrarily large inside the feasible set, and
- if it is a minimization problem, there might be a minimum, or it might be possible to make the objective arbitrarily small (big and negative) inside the feasible set.

If the feasible set is not bounded

If there **is** a maximum/minimum, it can happen

- uniquely at a corner,
- at two adjacent corners and at all points in between,
- at a corner or along an infinite ray leaving that corner, or
- along an entire infinite line.

This is a little more nuanced than the Theorem stated on page 239 of the text (which is not really a true theorem ☹).

The next slide present some examples.

Some unusual examples

Suppose the only constraint is $y \geq 0$. Then the feasible set is unbounded and has no corners.

- Objective $= y$ has no maximum. It has a minimum, reached along the entire x-axis.
- Objective $= -y$ has no minimum, but has a maximum
- Objective $= x - y$ has no minimum, and no maximum

Suppose the constraints are $y \geq 0$ and $x \geq 0$. Then the feasible set is unbounded and has one corner.

- Objective $= x + y$ has a minimum, reached uniquely at the corner.
- Objective $= y$ has a minimum, reached along the ray starting at the corner and moving to the right.

Suppose the constraints are $y \geq 0$, $x \geq 0$, $y \leq 2$. Then the feasible set is unbounded and has two corners.

- Objective $= x$ has a minimum, reached at both corners, and between the two corners.

A diet example

Mr. Carter eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Carter's breakfast should provide at least 480 calories but less than or equal to 700 milligrams of sodium. Mr. Carter would like to maximize the amount of protein in his breakfast mix.

	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

Let x denote the number of ounces of Cereal A that Mr. Carter has for breakfast and let y denote the number of ounces of Cereal B that Mr. Carter has for breakfast. We find that the set of constraints for x and y are

$$100x + 140y \geqslant 480, \quad 150x + 190y \leqslant 700, \quad x \geqslant 0, \quad y \geqslant 0$$

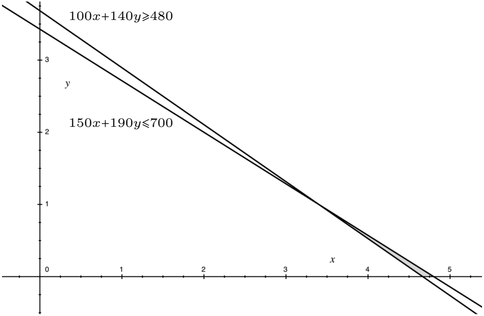
A diet example

(a) What is the objective function?

Objective = $9x + 10y$.

(b) Graph the feasible set.

$$100x + 140y \geqslant 480, \quad 150x + 190y \leqslant 700, \quad x \geqslant 0, \quad y \geqslant 0$$



The feasible set is the skinny triangle just above the x-axis.

A diet example

(c) Find the corners of the feasible set and the maximum of the objective function on the feasible set.

The two corners on the x-axis are $(4.8, 0)$ and $\left(\frac{16}{3}, 0\right)$. The intersection of $100x + 140y = 480$ and $150x + 190y = 700$ is the point $\left(\frac{17}{5}, 1\right)$. The values of the objective function are 42.2, 48 and $\frac{203}{5} = 40.6$. Hence the maximum of the objective function is 48 and it occurs at the point $\left(\frac{16}{3}, 0\right)$ on the boundary of the feasible region and nowhere else. (The minimum occurs at 40.6 where the two lines intersect).

(d) Conclusion?

To maximize intake of protein while keeping sodium and calories at acceptable levels, Mr. Carter should eat 5.33... ounces of Cereal A and none of cereal B.

Michael and the firefighter exam

Michael is taking an exam in order to become a volunteer firefighter. The exam has 10 essay questions and 50 short questions. He has 90 minutes to take the exam. The essay questions are worth 20 points each and the short questions are worth 5 points each. An essay question takes 10 minutes to answer and a short question takes 2 minutes. Michael must do at least 3 essay questions and at least 10 short questions. Michael knows the material well enough to get full points on all questions he attempts and wants to maximize the number of points he will get.

Michael and the firefighter exam

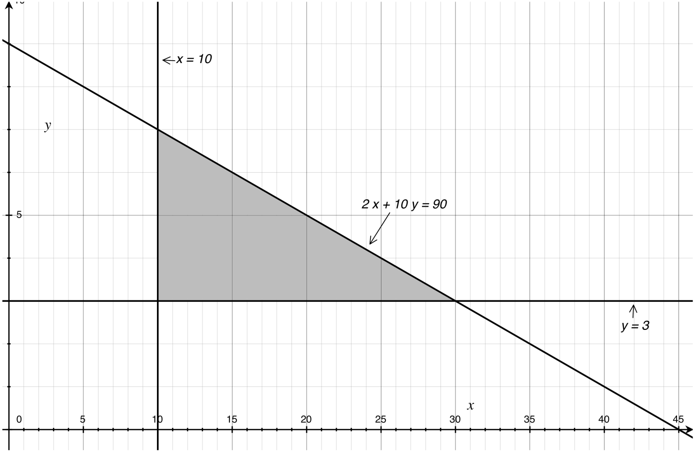
Let x denote the number of short questions that Michael will attempt and let y denote the number of essay questions that Michael will attempt. Write down the constraints and objective function in terms of x and y and find the/a combination of x and y which will allow Michael to gain the maximum number of points possible.

$2x + 10y \leqslant 90$ (time needed to answer the questions).
$x \geqslant 10$ (at least 10 short questions).
$x \leqslant 50$ (at most 50 short questions).
$y \geqslant 3$ (at least 3 essay questions).
$y \leqslant 10$ (at least 3 essay questions).
$5x + 20y$ is the objective function (Michael's total score).
It is required to maximize the objective.

Michael and the firefighter exam

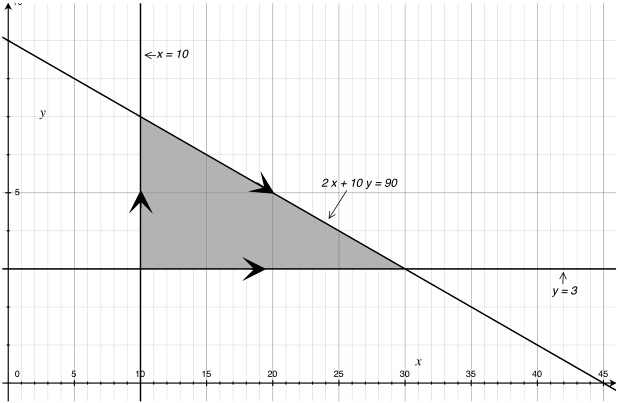
Here is the feasible region.

The lines $x = 50$ and $y = 10$ are outside the part of the plane shown (they are redundant)



Michael and the firefighter exam

The three corners are $(10, 3)$, $(10, 7)$ and $(30, 3)$. The values of the objective function at these corners are 60, 190 and 210. Hence Michael can maximize his score by answering 3 essay questions and 30 short questions.



Example with unbounded region

A local politician is budgeting for her media campaign. She will distribute her funds between TV ads and radio ads. She has been given the following advice by her campaign advisers;

- She should run at least 120 TV ads and at least 30 radio ads.
- The number of TV ads she runs should be at least twice the number of radio ads she runs but not more than three times the number of radio ads she runs.

The cost of a TV ad is \$8000 and the cost of a radio ad is \$2000. Which combination of TV and radio ads should she choose to minimize the cost of her media campaign?

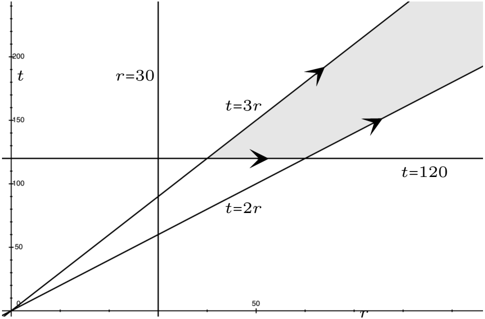
Let t be the number of TV ads and r the number of radio ads.

$$t \geqslant 120 \quad r \geqslant 30 \quad 2r \leqslant t \leqslant 3r$$

The objective function is $8000t + 2000r$ which she wishes to minimize.

Example with unbounded region

Here is the feasible region, which we see is unbounded.



If we wanted to *maximize* the campaign cost, we have no trouble; buy r radio ads and anywhere between $2r$ and $3r$ television ads, for a cost of between \$6,000r and \$8,000r — and r can be as large as we like.

The minimum cost occurs at the vertex where $t = 120$ and $t = 3r$ meet. This is the point $r = 40$, $t = 120$. Hence she should buy 40 radio ads and 120 TV ads.

Example with no feasible region

Mr. Baker eats a mix of Cereal A and Cereal B for breakfast. The amount of calories, sodium and protein per ounce for each is shown in the table below. Mr. Baker's breakfast should provide at least 600 calories but less than 700 milligrams of sodium. Mr. Baker would like to maximize the amount of protein in his breakfast mix.

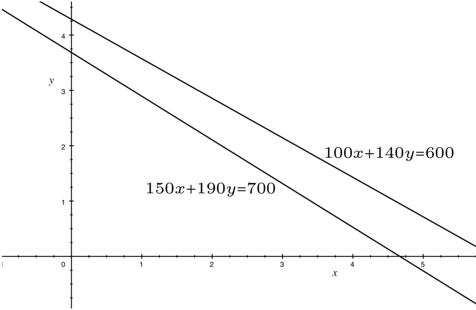
	Cereal A	**Cereal B**
Calories(per ounce)	100	140
Sodium(mg per ounce)	150	190
Protein(g per ounce)	9	10

Let x denote the number of ounces of Cereal A that Mr. Baker has for breakfast and let y denote the number of ounces of Cereal B that Mr. Baker has for breakfast.

Example with no feasible region

(a) Show that the feasible set is empty here, so that there is no feasible combination of the cereals for Mr. Baker's breakfast.

$100x + 140y \geqslant 600$ (calories)
$150x + 190y < 700$ (sodium)
$x \geqslant 0,\ y \geqslant 0$



Because we have $x \geqslant 0$, $y \geqslant 0$ we need only look at regions in the first quadrant.
Since $(0,0)$ satisfies $150x + 190y < 700$, $\mathrm{P}_0 = \{1\}$.
Since $(0,0)$ satisfies $100x + 140y < 600$, $\mathrm{P}_1 = \varnothing$.

Example with no feasible region

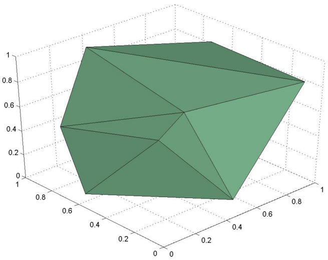
(b) If Mr. Baker goes shopping for new cereals, what should he look for on the chart giving the nutritional value, so that he can have some feasible combination of the cereals for breakfast?

This is an essay question with no single right answer. Basically Mr. Baker needs to choose cereals with either more calories or less sodium per ounce.

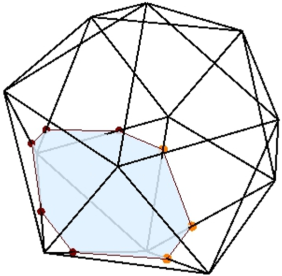
Linear programming with more variables

We've looked at problems with two variables, where the feasible set can be drawn in the plane, and with just a few constraints, so it is easy to find all the corner points, and see which one optimizes the objective.

If a problem has three variables, the feasible set can still be visualized, this time as a *polyhedron* — a shape in space bounded by flat polygons.



Linear programming with more variables



Again the optimum will occur at some corner point, where a plane of points with the same objective value crosses through the feasible region for the last time.

Problems with three variables and not too many constraints can be dealt with fairly easily by locating all the corner points, and seeing which gives the best value.

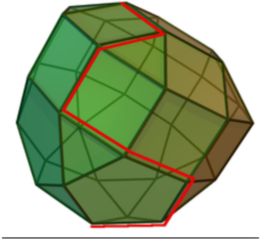
Linear programming with more variables

In the real world, linear programming problems arise frequently, whenever limited resources (planes, machines, people, ...) have to be allocated, subject to constraints (physical, legislative, consumer driven, ...), in such a way as to maximize something (revenue, number of passengers moved, ...) or minimize something (cost, time, ...).

Most linear programming problems have more than three variables. For example, one benchmarking problem used to compare computer programs that solve these problems has 428,032 variables and 986,069 constraints (see `http://plato.asu.edu/bench.html`)

Linear programming with more variables

The *Simplex Algorithm* is an efficient way of solving problems with many variables. It starts by finding *any* corner point of the feasible set (not necessarily the best), then it looks at nearby corner points, and moves to one that gives a better value for the objective function. In this way it climbs quickly to the corner point with the best objective value, without having to figure out first what *all* the corner points are.



Linear programming with more variables

The Simplex Algorithm was invented by George Dantzig (the actual protagonist in the "unsolvable math problem" urban legend, `http://www.snopes.com/college/homework/unsolvable.asp`) in 1947, and was an important part of the post-WWII industrial boom.