# Linear Inequalities in Two Variables

The next topic we will study is *optimization* — how to make the most of limited resources. This will lead us to situations like the following: if apples cost \$1.45 per kilo and pears cost \$1.25 a kilo, what combination of apples and pears can I buy with at most \$5?

If I buy $a$ kilos of apples and $p$ kilos of pears then I spend $1.45a$ on apples and $1.25p$ on pears, so $1.45a + 1.25p$ in total. So whatever combination I buy, it must satisfy

$$1.45a + 1.25p \leq 5.$$

This is an example of a *linear inequality*.

# Review of lines

The equation of a line is given by:

$$\boxed{ax + by = c.}$$

for some given numbers $a, b$ and $c$.
A vertical line which runs through the point $c$ on the x-axis has equation

$$\boxed{x = c.}$$

A horizontal line which runs through the point $d$ on the y-axis has equation

$$\boxed{y = d.}$$

A line which runs through the point $(0, 0)$ has an equation of the form

$$\boxed{ax + by = 0.}$$

# Review of lines

Minor technical issue: if $a = b = 0$ then either

- all points satisfy the equation (if $c = 0$) or
- no points satisfy the equation (if $c \neq 0$).

Whenever we discuss *the line* $ax + by = c$ we agree that not both $a$ and $b$ are 0.

Given an equation of a line, its graph is the set of all points in the $xy$-plane which satisfy the equation.

In particular the graph is an example of a set and we can form unions, complements, intersections, etc.

From plane geometry you know that the intersection of two lines is either the empty set (the lines are parallel), or the line (the lines are equal) or a single point.

You can find this single point (if it exists) with a little algebra.

# Review of lines

**Example**: The line $2x + 3y = 6$.

The point $(0, 2)$ satisfies the equation $2x + 3y = 6$, because $2(0) + 3(2) = 6$. Hence the point $(0, 2)$ is on the graph of that equation, so is on the line.

The point $(0, 0)$ does not satisfy the equation $2x + 3y = 6$, since $2(0) + 3(0) \neq 6$. Hence the point $(0, 0)$ is not on the graph of the equation $2x + 3y = 6$, and is not on the line.

# Review of lines

We can draw the graph of a line if we know the location of any 2 points on the line. The $x$- and $y$-intercepts (the places where the line hits the $x$-axis and the $y$-axis) are usually the easiest points to find.

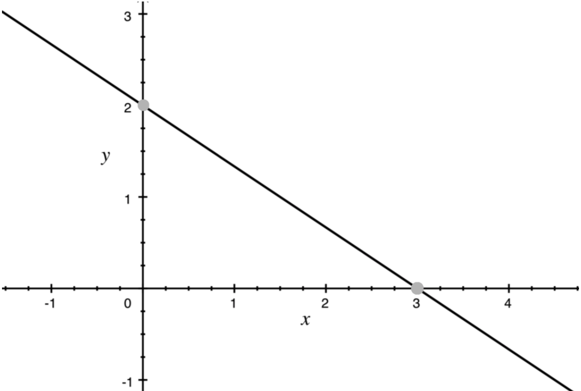
- To find the $x$-intercept, we set $y = 0$ in the equation and solve for $x$.
- To find the $y$-intercept, we set $x = 0$ in the equation and solve for $y$.

Given these two points (or any two distrint points) we can draw the line by joining the points with a straight edge and extending.

# Review of lines

**Example** Find the $x$- and $y$- intercept of the line with equation $2x + 3y = 6$ and draw its graph.

- The $x$-intercept occurs when $y = 0$: hence $2x = 6$ so $x = 3$.
- The $y$-intercept occurs when $x = 0$: hence $3y = 6$ so $y = 2$.



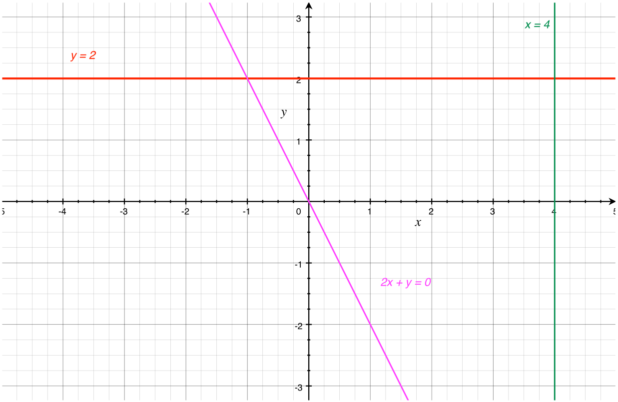
# Review of lines

**Graphing lines with only one intercept.** There are three situations where a line has only one intercept:

- The graph of an equation of the form $x = c$ is a vertical line which cuts the $x$-axis at $c$.
- The graph of an equation of the form $y = d$ is a horizontal line which cuts the $y$-axis at $d$.
- The graph of an equation of the form $ax + by = 0$ cuts both axes at the point $(0, 0)$, so one needs to pick another value of $x$ (or $y$), find the corresponding value of $y$ (or $x$) with some algebra, and plot the corresponding point.

# Review of lines

**Example** Draw the graphs of the lines $y = 2$, $x = 4$ and $2x + y = 0$.




To draw $2x + y = 0$ note $(0, 0)$ is one point. Pick any non-zero value for $x$ and solve for $y$; if $x = -1$, $y = 2$.

# Review of lines

**Problem**: Given two points $(x_0, y_0)$ and $(x_1, y_1)$, what is the equation of the line that passes through the two points?

**Solution**: Here is one way to get the equation.

1. Find $a$ and $b$ by looking at the difference of the two points: $(x_1, y_1) - (x_0, y_0) = \big(x_1 - x_0, y_1 - y_0\big) = (b, -a)$.
2. Find $c$ by solving $c = ax_0 + by_0$.
3. The equation of the line is $ax + by = c$.

**Check**:

- Is the point $(x_0, y_0)$ on this line? YES, because $c = ax_0 + by_0$ (from Step 2)
- Is the point $(x_1, y_1)$ on this line? YES, because $c = ax_1 + by_1$ is the same as $ax_0 + by_0 = ax_1 + by_1$ (from Step 2), which is the same as $a(x_1 - x_0) + b(y_1 - y_0) = 0$, which is the same as $ab + b(-a) = 0$ (by Step 1), which is a correct equation.

# Review of lines

**Example**: Find the equation of the line through $(1, 2)$ and $(5, 7)$.

**Solution**:

1. $(5, 7) - (1, 2) = (4, 5)$ so $a = -5, b = 4$
2. $-5x_0 + 4y_0 = -5 \cdot 1 + 4 \cdot 2 = 3$ so $c = 3$
3. The equation of the line is $\boxed{-5x + 4y = 3.}$

**Example**: Find the equation of the line through $(1, 0)$ and $(5, 0)$.

**Solution**:

1. $(5, 0) - (1, 0) = (4, 0)$ so $a = 0, b = 4$
2. $0x_0 + 4y_0 = 0 \cdot 1 + 4 \cdot 0 = 0$ so $c = 0$
3. The equation of the line is $\boxed{4y = 0.}$ (or $\boxed{y = 0.}$)

# Linear Inequalities in Two Variables

To solve some optimization problem, specifically *linear programming* problems, we must deal with **linear inequalities** of the form

$$\begin{aligned} ax + by &\geqslant c \\ ax + by &\leqslant c \\ ax + by &> c \\ ax + by &< c, \end{aligned}$$

where $a$, $b$ and $c$ are given numbers. Constraints on the values of $x$ and $y$ that we can choose to solve our problem, will be described by such inequalities.

# Linear Inequalities in Two Variables

**Example**: Michael is taking a exam to become a volunteer firefighter. The exam has 10 essay questions and 50 short questions. Michael has 90 minutes to take the exam and knows he is not expected to answer every question. An essay question takes 10 minutes to answer and a short question takes 2 minutes. Let $x$ denote the number of short questions that Michael will attempt and let $y$ denote the number of essay questions that Michael will attempt. What linear inequalities describes the constraints on Michael's time given above?

There is a time constraint: $\boxed{2x + 10y \leqslant 90}$.

Additionally, since Michael can't answer a negative number of questions, there are constraints $\boxed{x \geqslant 0}$ and $\boxed{y \geqslant 0}$. Furthermore, since he can not answer more questions than there are, so $\boxed{x \leqslant 50}$ and $\boxed{y \leqslant 10}$.

# Language of linear inequalities

- A point $(x_1, y_1)$ is said to **satisfy** the inequality $ax + by < c$ if $ax_1 + by_1 < c$.
- It satisfies $ax + by > c$ if $ax_1 + by_1 > c$.
- It satisfies $ax + by \leqslant c$ if either $ax_1 + by_1 < c$ or $ax_1 + by_1 = c$.
- It satisfies $ax + by \geqslant c$ if either $ax_1 + by_1 > c$ or $ax_1 + by_1 = c$.

The **graph** of a linear inequality is the set of all points in the plane which satisfy the inequality.

Notice that any point $(x_1, y_1)$ satisfies **exactly one** of $ax + by > c$, $ax + by < c$ or $ax + by = c$.

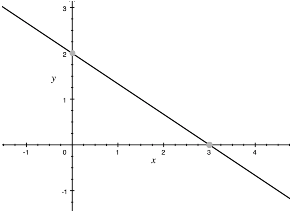
# Example

Determine if the point $(x, y) = (1, 2)$ satisfies the inequality $2x + 3y \geqslant 6$.

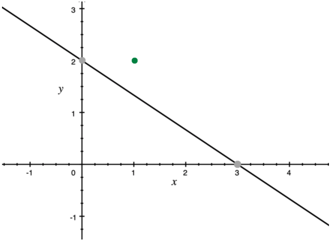
$2 \cdot 1 + 3 \cdot 2 = 8 > 6$ so yes $(1, 2)$ satisfies the inequality.

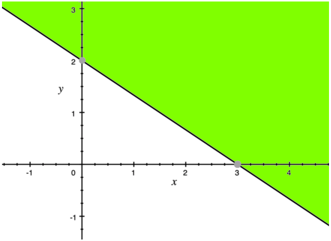
Shade all the points which satisfy $2x + 3y \geqslant 6$.

Draw the line $2x + 3y = 6$



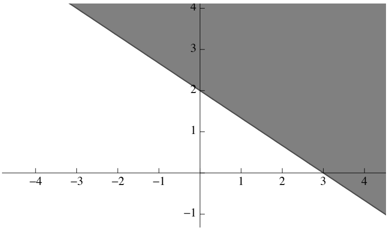

Plot $(1, 2)$, shade all points on same side of line.

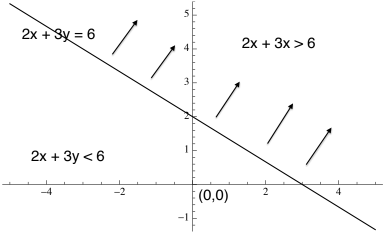


# Example

We can represent the graph of the inequality either by shading or with arrows:



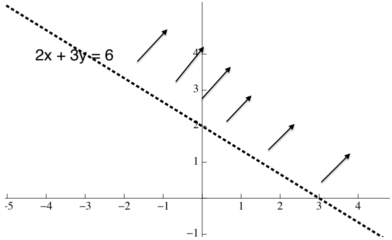

The plot with arrows will be more useful when we want to plot many inequalities simultaneously.

# $>$ versus $\geq$

The points satisfying an inequality like $2x + 3y \geq 6$ include all points on the line $2x + 3y = 6$. We indicate this by drawing the graph of this line solidly.

We use a *d*otted line when we work with a strict inequality like $2x + 3y > 6$:

**Graph of the inequality**
$2x + 3y > 6$



# More language of inequalities

Any line divides the plane into two disjoint subsets called *half-planes*.

- If the line is not vertical, there is an *upper half-plane* and a *lower half-plane*.
- If the line is not horizontal, there is a *right half-plane* and a *left half-plane*.
- If the line is neither vertical or horizontal then
  - sometimes right half-plane equals upper half plane
  - sometimes left half-plane equals upper half-plane
- Using our set theory terminology, the union of the two half-planes is the complement of the line.

Once you draw the line, it should be easy to pick out the upper/right and lower/left half-planes.

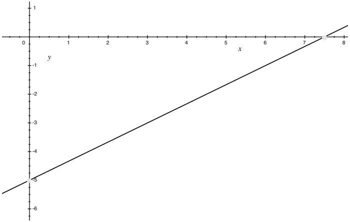
# Graphing an inequality

To graph an inequality of the form $ax + by \leqslant c$:

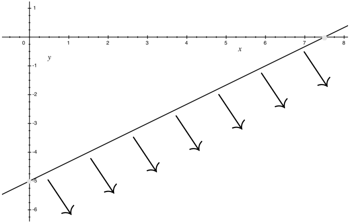
- first draw the line $ax + by = c$.
- One of the two resulting half-planes is the solution set for $ax + by > c$, and the other is the solution set for $ax + by < c$.
- To decide which is which, pick a test point $(x_1, y_1)$ in one of the half-planes and see which inequality holds.
  - If $ax_1 + by_1 < c$ then all points on the same side of the line as $(x_1, y_1)$ satisfy the inequality.
  - If $ax_1 + by_1 > c$ then all points on the opposite side of the line to $(x_1, y_1)$ satisfy the inequality.
- Draw in the correct half-plane by shading or with arrows.
- Same technique works for $ax + by < c$, $ax + by \geq c$ and $ax + by \geq c$.
- If $ax + by = c$ doesn't pass through $(0, 0)$, $(0, 0)$ is an excellent choice of test point.

# Example: $2x - 3y \geqslant 15$

Here's the graph of $2x - 3y = 15$:

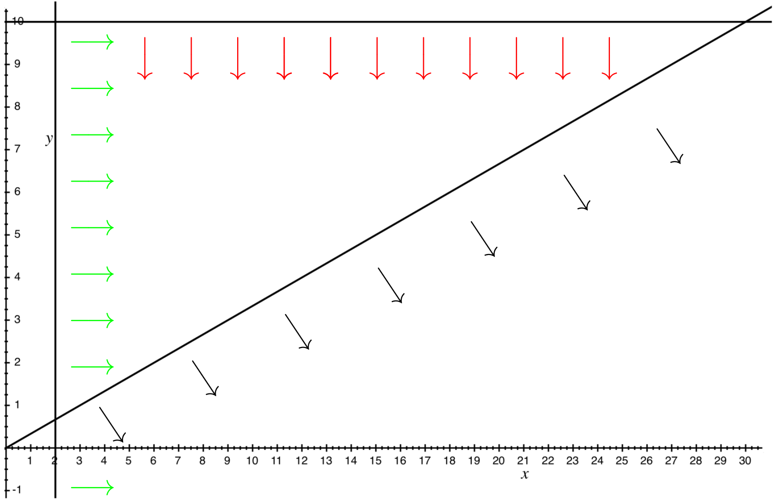



At $(0, 0)$, $2x - 3y = 0 < 15$ which is less than 15. So we need to shade the side of the line opposite $(0, 0)$:



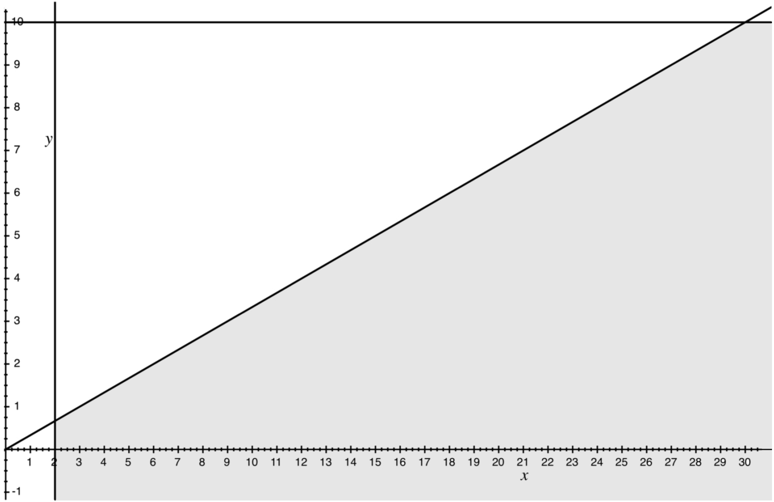
# Example: $x - 3y \geqslant 0, \quad x > 2, \quad y \leqslant 10$

Here's an example where we want to satisfy three inequalities simultaneously. First we draw each of the three inequalities independently, but on the same graph.

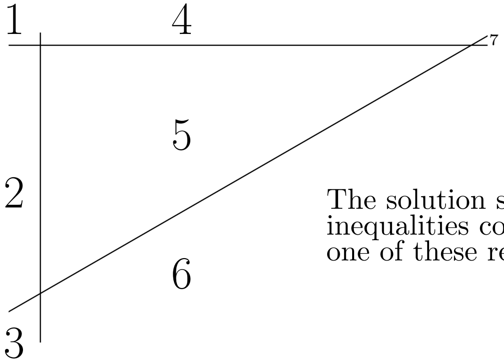


## Example: $x - 3y \geqslant 0, \quad x > 2, \qquad y \leqslant 10$

Next, we shade in that part of the plane that *simultaneously* satisfies *all three* inequalities:



# Dealing with many inequalities in general

- Draw all the lines corresponding to the inequalities.
- Ignore the axes *unless* they are explicitly some of the lines.
- Identify the regions into which the plane is divided. Some regions will be infinite, so you only see a small part of them.
- 

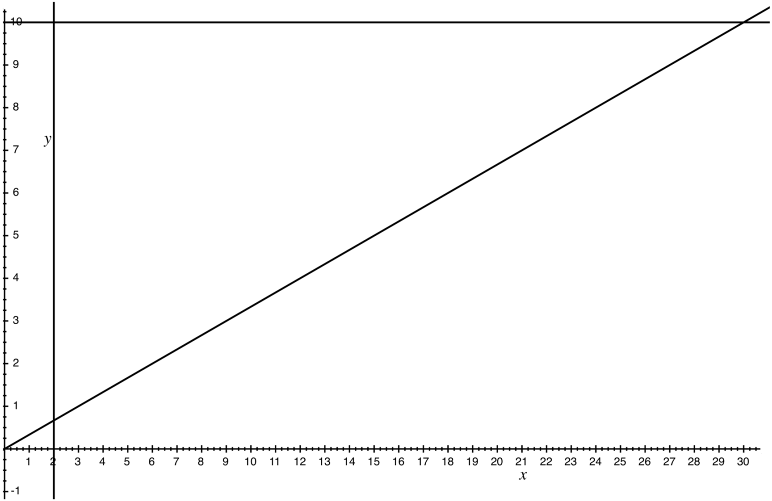

The solution set of your inequalities consists of exactly one of these regions

# Dealing with many inequalities in general

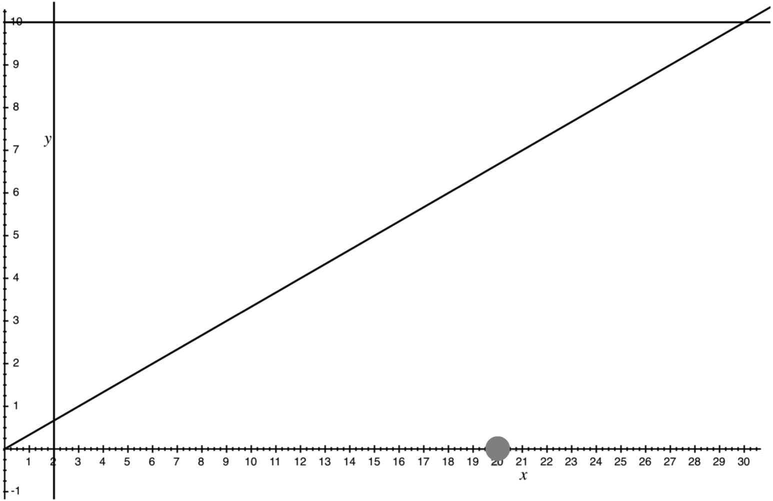
- Add the axes back in.
- Pick a point in the region you think is the solution set.
- Check that your pick satisfies all the inequalities. If it doesn't, you need to identify another region as the possible solution set.
- Once you have found the right region, and tested it using a test point, shade the region containing your point.

For the example $x - 3y \geqslant 0, x > 2, y \leqslant 10$, this process is illustrated on the next three slides.

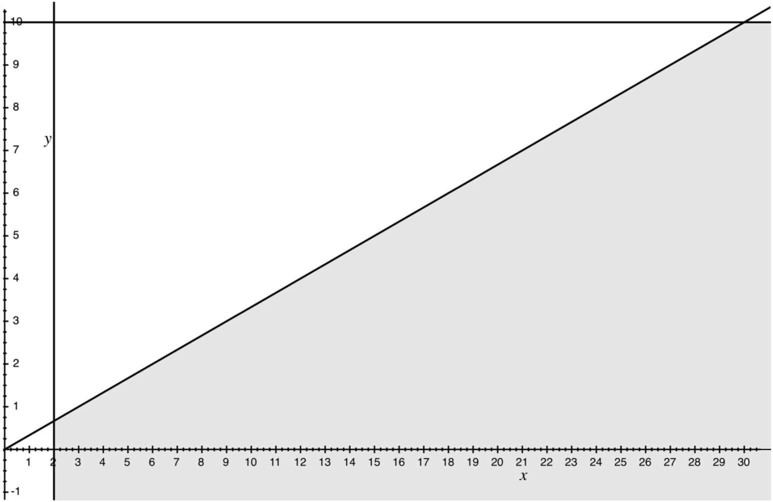
# Example: $x - 3y \geqslant 0, x > 2, y \leqslant 10$



# Example: $x - 3y \geqslant 0, x > 2, y \leqslant 10$



# Example: $x - 3y \geqslant 0, x > 2, y \leqslant 10$



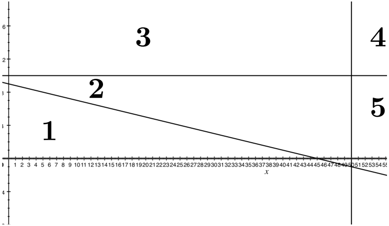
# Example

Returning to a previous example: Michael is taking a exam to become a volunteer firefighter. The exam has 10 essay questions and 50 short questions. Michael has 90 minutes to take the exam and knows he is not expected to answer every question. An essay question takes 10 minutes to answer and a short question takes 2 minutes. Let $x$ denote the number of short questions that Michael will attempt and let $y$ denote the number of essay questions that Michael will attempt. Here are the linear inequalities describing the constraints on Michael:

- Time constraint $\boxed{2x + 10y \leqslant 90}$.
- $\boxed{x \geqslant 0}$ and $\boxed{y \geqslant 0}$.
- $\boxed{x \leqslant 50}$ and $\boxed{y \leqslant 10}$.

# Example

Here are the lines $2x + 10y = 90$, $x = 0$, $y = 0$, $x = 50$, $y = 10$.




- We've only numbered regions in the *first quadrant* of the plane where $x \geq 0$, $y \geq 0$ since these two constraints must be satisfied. This saves a lot of labor!
- $x \leqslant 50$ and $y \leqslant 10$ are redundant — we can't have $2x + 10y \leqslant 90$ if $x > 50$ or $y > 10$. But you often can't tell if a constraint is redundant until you draw the graph, so it's best to err on the side of caution and graph too many constraints rather than graph too few.

# Example

We should suspect that region **1** is the one region where all five constraints are satisfied simultaneously. Unfortunately we can't use $(0, 0)$ as a test point, since it lies on the constraint lines $x = 0$ and $y = 0$. But we can use the point $(1, 1)$, which is definitely inside region **1**

Check that $(1, 1)$ satisfies all five constraints:

- $2(1) + 10(1) = 12 \leq 90$
- $1 \geq 0$, $1 \geq 0$
- $1 \leq 50$, $1 \leq 10$

This shows that the solution set to the system of five inequalities is region **1**. Since all the inequalities are $\leqslant$, we draw the constraint lines as solid lines.

# Example

The solution: