Intuition test

Around one million people in the U.S. (population around 308 million) have a certain particularly nasty condition, condition X. It's a condition with no obvious external symptoms, and susceptibility can't be inferred from medical or family history. It strikes at random.

A test exists for X, which is 95% accurate — the test correctly identifies the presence of X 95% of the time that it is present, and correctly identifies the absence of X 95% of the time that is not present.

Being a hypocondriac, I have myself tested for X, and the test comes back positive. What is the probability that I have X?

A: more than 90%

B: around 75%

C: around 50%

D: around 25%

E: less than 10%

Bayes' Theorem

Now we look at how we can use information about conditional probabilities to calculate reverse conditional probabilities — i.e., how we calculate $\mathbf{P}\left(A \middle| B\right)$ when we know $\mathbf{P}\left(B \middle| A\right)$ (and some other things). We will generally solve these problems with tree diagrams, but we will also see a formula that says what's going on algebraically.

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Example Suppose that a factory has two machines, Machine A and Machine B, both producing jPhone touch screens. 40% of production is from Machine A and 60% is from Machine B. 10% of the touch screens produced by Machine A are defective and 5% of those from Machine B are.

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If I randomly choose a touch screen produced in the factory, then there is a 40% probability that it came from Machine A.

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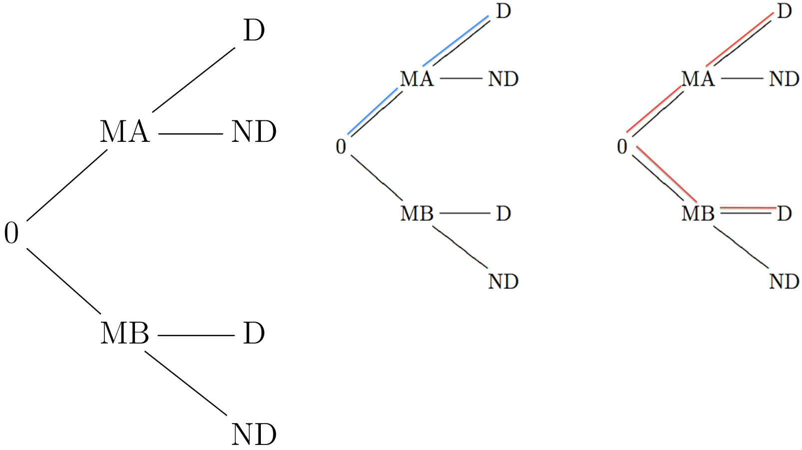
Suppose that I test the randomly chosen screen, and find that it is defective, now what is the probability that it came from Machine A? Greater or less than 40%?

Factory example

We can draw a tree diagram representing the information we are given. If we choose a touch screen at random from those produced in the factory, we let MA be the event that it came from Machine A and let MB be the event that it came from Machine B. We let D denote the event that the touch screen is defective and let ND denote the event that it is not defective. Fill in the appropriate probabilities on the tree diagram on the left on the next page.

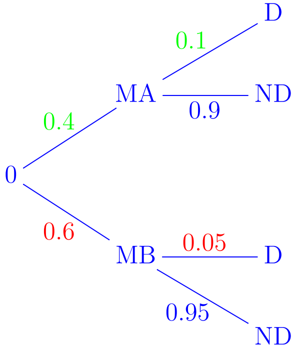
Factory example



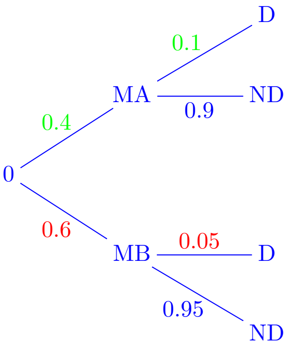
Factory example

We can now calculate $\mathbf{P}\left(MA \middle| D\right) = \dfrac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)} =$

$$\frac{\mathbf{P}(MA \cap D)}{\mathbf{P}\left(D \middle| MA\right) \cdot \mathbf{P}(MA) + \mathbf{P}\left(D \middle| MB\right) \cdot \mathbf{P}(MB)}.$$ Note the event D is shown in red above and the event $MA \cap D$ is shown in blue.



Factory example



$$\mathbf{P}\left(MA \middle| D\right) = \frac{\mathbf{P}(MA \cap D)}{\mathbf{P}(D)} =$$

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$$\frac{0.4 \cdot 0.1}{0.4 \cdot 0.1 + 0.6 \cdot 0.05} = \frac{0.04}{0.07} = \frac{4}{7} \approx 57\% > 40\% = \mathbf{P}(MA)$$

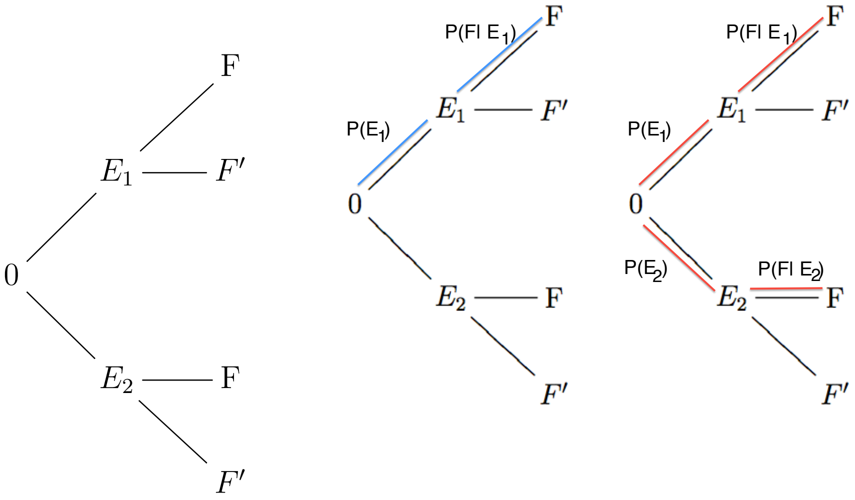
Bayes' Theorem

Let E_1 and E_2 be mutually exclusive events $(E_1 \cap E_2 = \emptyset)$ whose union is the sample space, i.e. $E_1 \cup E_2 = S$. Let F be an event in S for which $\mathbf{P}(F) \neq 0$. Then

$$\boxed{\begin{aligned}\mathbf{P}\left(E_1 \middle| F\right) = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} &= \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F)} = \\ &\frac{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right)}{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right) + \mathbf{P}(E_2)\mathbf{P}\left(F \middle| E_2\right)}.\end{aligned}}$$

Note that if we cross-classify outcomes in the sample space according to whether they belong to E_1 or E_2 and whether they belong to F or F', we get a tree diagram as above from which we can calculate the probabilities.

Bayes' Theorem



A more general Bayes' Theorem

Let $E_1, E_2, \ldots, E_n$ be (pairwise) mutually exclusive events such that $E_1 \cup E_2 \cup \cdots \cup E_n = S$, where S denotes the sample space. Let F be an event such that $\mathbf{P}(F) \neq 0$, Then

$$\mathbf{P}\left(E_1 \middle| F\right) = \frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(F)} =$$

$$\frac{\mathbf{P}(E_1 \cap F)}{\mathbf{P}(E_1 \cap F) + \mathbf{P}(E_2 \cap F) + \cdots + \mathbf{P}(E_n \cap F)} =$$

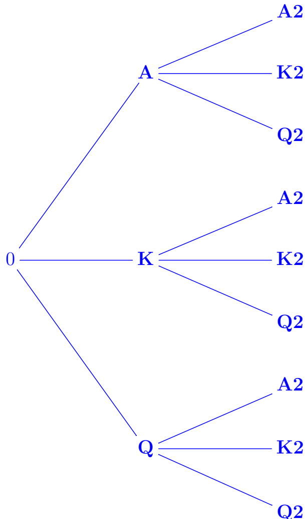
$$\frac{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right)}{\mathbf{P}(E_1)\mathbf{P}\left(F \middle| E_1\right) + \mathbf{P}(E_2)\mathbf{P}\left(F \middle| E_2\right) + \cdots + \mathbf{P}(E_n)\mathbf{P}\left(F \middle| E_n\right)}$$

A more general Bayes' Theorem

Example A pile of 8 playing cards has 4 aces, 2 kings and 2 queens. A second pile of 8 playing cards has 1 ace, 4 kings and 3 queens. You conduct an experiment in which you randomly choose a card from the first pile and place it on the second pile. The second pile is then shuffled and you randomly choose a card from the second pile. If the card drawn from the second deck was an ace, what is the probability that the first card was also an ace?

A more general Bayes' Theorem

Let **A** be the event that you draw an ace, **K** the event that you draw a king and **Q** be the event that you draw a queen.



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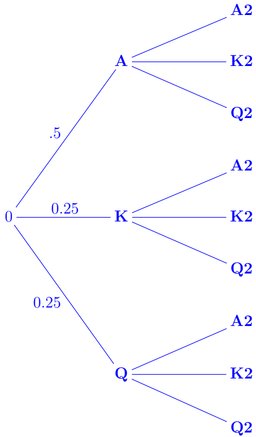
In the first round there are $4+2+2=8$ cards so the probabilities in the first round are



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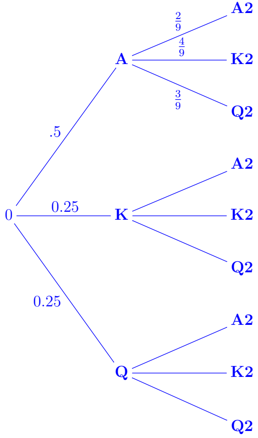
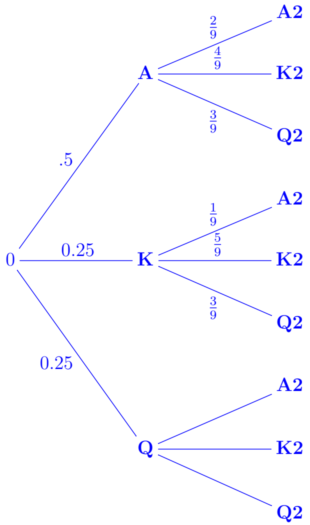
In the second round there are $1 + 4 + 3 + 1 = 9$ cards and the probabilities are different at the various nodes. If you draw an ace in round 1 the cards are 2 aces, 4 kings and 3 queens so we get



A more general Bayes' Theorem

If you draw a king in round 1 the cards are 1 ace, 5 kings and 3 queens so we get



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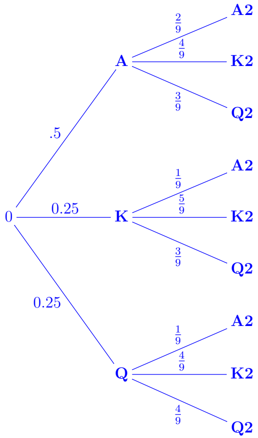
If you draw a queen in round 1 the cards are 1 ace, 4 kings and 4 queens so we get



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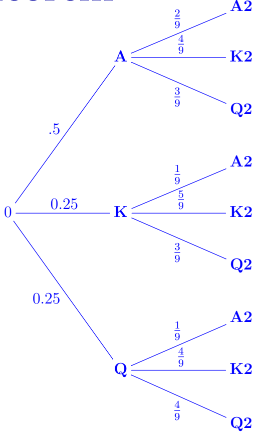

The question asks for

$$\mathbf{P}\left(A \middle| A2\right) = \frac{\mathbf{P}(A \cap A2)}{\mathbf{P}(A2)} = \frac{\frac{1}{2} \cdot \frac{2}{9}}{\frac{1}{2} \cdot \frac{2}{9} + \frac{1}{4} \cdot \frac{1}{9} + \frac{1}{4} \cdot \frac{1}{9}} = \frac{2}{3}.$$

Predictive value of diagnostic test

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Example: Around one million people in the US (population around 308 million) have a certain particularly nasty condition, condition X. A test exists for X, which is 95% accurate — the test correctly identifies the presence of X 95% of the time, and correctly identifies the absence of X 95% of the time.

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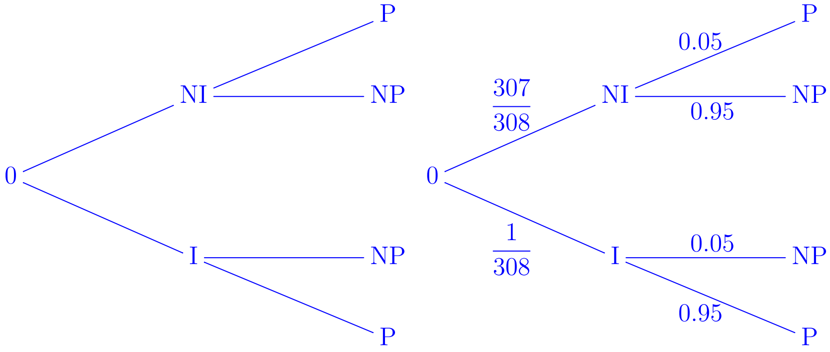
Being a hypocondriac, I have myself tested for X, and the test comes back positive. What is the probability that I have X? That is, letting P denote the event that a person chosen at random from the population tests positive, and letting I denote the event that a person chosen at random has X, what is $\mathbf{P}\left(I \middle| P\right)$?

Predictive value of diagnostic test



$$\mathbf{P}\left(I \middle| P\right) \approx \frac{\frac{1}{308}\cdot 0.95}{\frac{307}{308}\cdot 0.05 + \frac{1}{308}\cdot 0.95} \approx \frac{0.00308}{0.0498 + 0.00308} \approx$$

$$\frac{0.00308}{0.0529} \approx 0.0583 = 5.83\%$$

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In country C, 40% of the residents have condition X. Suppose a random resident of Country C tests positive for X. What is the probability that the person actually has X?

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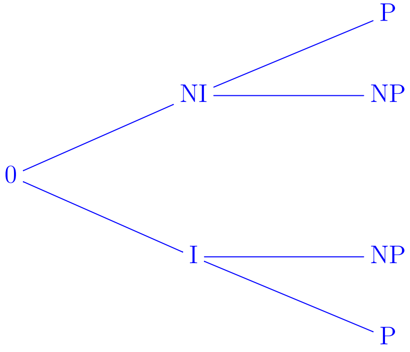

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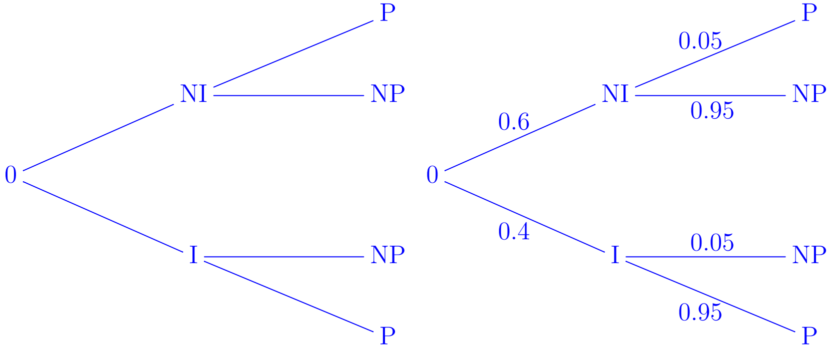

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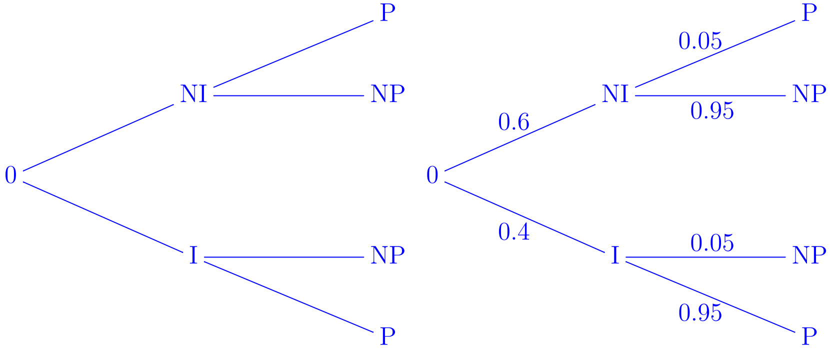

$$\mathbf{P}\left(I \middle| P\right) = \frac{0.4 \cdot 0.95}{0.6 \cdot 0.05 + 0.4 \cdot 0.95} \approx 93\%$$

Predictive value of diagnostic test

Example A test for Lyme disease is 60% accurate when a person has the disease and 99% accurate when a person does not have the disease. In Country Y, 0.01% of the population has Lyme disease. What is the probability that a person chosen randomly from the population who test positive for the disease actually has the disease?

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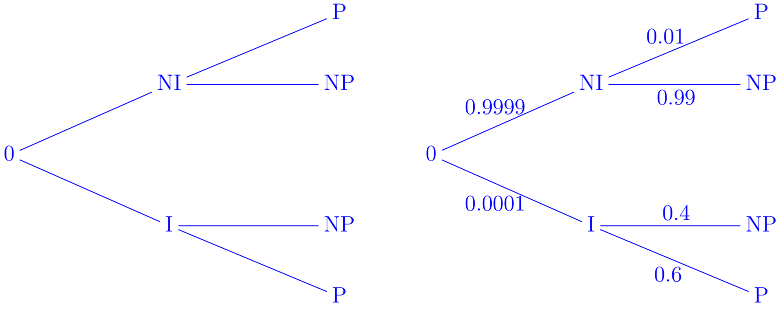

$$\mathbf{P}\left(I\middle|P\right) = \frac{0.0001 \cdot 0.06}{0.9999 \cdot 0.01 + 0.0001 \cdot 0.06} \approx 0.006.$$

A legal example

A crime has been committed and the only evidence is a blood spatter that could only have come from the perpetrator. The chance of a random individual having the same blood type as that of the spatter is 10%. Joe has been arrested and charged. The trial goes as follows.
Prosecutor: Since there is only a 10% chance that Joe's blood would match, there is a 90% chance that Joe did it. That's good enough for me.
Defence Lawyer: There are two hundred people in the neighborhood who could have done the crime. Twenty of them will have the same blood type as the sample. Hence the chances that Joe did it are $\frac{1}{20} = 5\%$ so there is a 95% chance that Joe is innocent. That's good enough for me.

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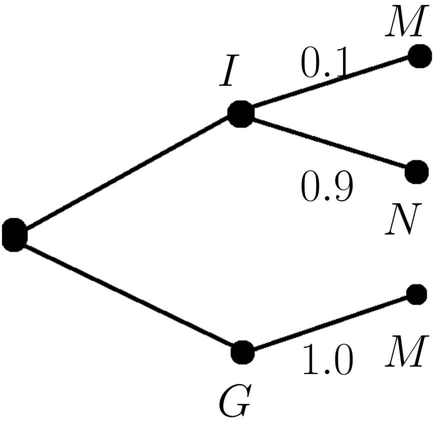

If $\mathbf{P}(I) = x$ and so $\mathbf{P}(G) = 1 - x$ then

$$\mathbf{P}\left(I \middle| M\right) = \frac{0.1 \cdot x}{0.1 \cdot x + 1 \cdot (1 - x)} = \frac{0.1x}{1 - 0.9x}$$

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If you initially think that there is a 40% chance that Joe is guilty, then $x = 0.4$ and after seeing the evidence $\mathbf{P}\left(I \middle| M\right) = 0.0625$.

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If you suspect that the police searched a blood-type database until they came up with a name in the neighborhood, then you might initially think that $x = P(I) = \dfrac{19}{20} = 95\%$. Now after seeing the evidence Bayes suggests revising to $\mathbf{P}\left(I \middle| M\right) = 0.66$.

Some resources

Here's an article on the predictive value of diagnostic tests:

`Doctors flunk quiz on screening-test math`

If you are more legally inclined, here is a discussion of Bayes Theorem as it applies to criminal trials.