Sets

A **set** is a collection of objects. The objects are called **elements** of the set.

A set can be described as a list, for example $D = \{5, 6, 7\}$ or with words (often in many different ways):

$$\begin{aligned} D &= \{\text{All whole numbers between 5 and 7 inclusive}\} \\ &= \{\text{All integers bigger than 4 and less than 8}\} \end{aligned}$$

Repetitions in the list or changes in the order of presentation do not change the set. For example, the following lists describe the same set:

$$\{5, 5, 6, 7\} = \{5, 6, 7\} = \{6, 5, 7\}.$$

Sets

When describing the elements of a set, we should be careful that there is **no ambiguity in our description** and the set is well defined.

For example, to talk about "the set of the 15 sexiest men of the twenty-first century" does not make sense, since the description is subject to personal opinion and different people may produce different sets from this description.

On the other hand, the **set** of "men named 'sexiest man alive' by *People* magazine between 2001 and 2015 (inclusive)" is unambiguous.

Notation

We read the notation "$5 \in D$" as "*5 is an element of D*". We read the notation "$2 \notin D$" as "*2 is not an element of D.*"

Equal Sets We say two sets are **equal** if they consist of exactly the same elements. For example consider the following sets
$A = \{$Major league baseball players who hit more than 760 home runs in their career$\}$,
$B = \{$Major league baseball players who hit more than 70 home runs in a single season$\}$.

These two sets are equal and have a single element. We have $A = B = \{$Barry Bonds$\}$.

Infinite Sets and dot notation

A set may have infinitely many elements, so we can't list all of them. For example let
$E = \{\text{all even integers greater than or equal to } 1\}$.

We write this as $E = \{2, 4, 6, \ldots\}$, where "$\ldots$" should be read as "*et cetera*".

When we place an element after the dots, as in
$K = \{2, 4, 6, \ldots, 100\}$, this indicates that we are talking about the finite set of even numbers greater than 1 and less than or equal to 100 (the last element on the list is 100).

NB: This notation assumes we have an implicit agreement as to the formula for the remaining terms. For example E might have been $\{2, 4, 6, 12, 14, 16, 22, 24, 26, \ldots\}$.

Set-Builder Notation

A description of the set D above may also be written using set-builder notation:

$$D = \{x |\ x \text{ is an integer between 5 and 7 inclusive}\}$$

or

$$D = \{x |\ x \text{ is an integer and } \ 5 \leq x \leq 7\}.$$

Here the symbol | is read as "such that" and the upper mathematical sentence above reads as "D is equal to the set of all x such that x is an integer between 5 and 7 inclusive".

The Empty Set

The empty set is the set with no elements, i.e. the list of its elements is a blank list. It is denoted by the symbol $\emptyset$. One can think of the empty set as an empty list: $\{\ \}$.

This set can have many verbal descriptions, for example:

$\{$all major league baseball players who got more than 80 home runs in a single season$\} = \emptyset$.

$\{$Years in which Jason Biggs won Best Actor Oscar$\} = \emptyset$.

Subsets

A **Subset** of a set A is a collection of elements of A. We have B is a subset of A, denoted by $B \subseteq A$, if every element of B is also an element of A. We say that B is a **proper subset** of A if $B \subseteq A$, but $B \neq A$.

Example If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$ and $D = \{2, 6\}$,

Is $D \subseteq A$? No: $6 \in D$ but $6 \notin A$.

Is $D \subseteq B$? Yes. $2 \in B$ and $6 \in B$ so every element in D is in B.

Is $D \subseteq C$? No: $2 \in D$ but $2 \notin C$.

Is D a proper subset of B? Yes: For example $8 \in B$ but $8 \notin D$.

Subsets

A itself is a subset of A, because it complies with the requirement in the definition above. A set is never a proper subset of itself.

The empty set is a subset of any set: $\emptyset \subseteq A$ for every set A. It is always proper unless $A = \emptyset$.

Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

The collection of all the subsets of a set is called the **power set**. For example, the power set of $\{a, b, c\}$ has eight elements: $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$

Universal Sets

Sometimes we wish to restrict our attention to a particular set, called a *universal set* and usually denoted by U. For example, in doing a voter polling survey you might want to restrict attention to likely voters. In this case the universal set would be likely voters (however you defined it). Or you might want to survey likely Republican voters which would give you a different universal set.

Universal Sets

Example If we do a survey on music preferences in our class, we would use $U = \{\text{all students in our class}\}$ as our universal set.

If we let $R = \{\text{students in the class who like Rap music}\}$,

$C = \{\text{students in the class who like Classical music}\}$,

and $E = \{\text{students in the class who like 80's music}\}$,

then R, C and E are all subsets of our universal set U.

Note To avoid ambiguity in the definition of such sets, it is common in surveys such as this to restrict answers to the given questions to "yes" and "no"; (e.g. "Do you like Rap music? Yes No (circle one)"). In doing this the resulting sets are well defined, but of course we have ignored the tastes of those who like some rap music but not all.

Union

Given two sets, A and B, we define their **union**, denoted $A \cup B$, to be the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Important note: when we say "or" we **always** mean "inclusive or". So if $a \in A$ and $a \in B$, then $a \in A \cup B$.

Notes:

- $A \subseteq A \cup B$ and $B \subseteq A \cup B$.
- $A \cup B = B \cup A$.
- $A \cup \emptyset = \emptyset \cup A = A$.

Union

Example If $A = \{1, 2, 6, 7, 8\}$ and $B = \{-1, 3, 6, 8\}$, what is $A \cup B$?

$A \cup B = \{1, 2, 6, 7, 8, -1, 3\}$.

Union

Union of 3 sets If A and B and C are sets, their union $A \cup B \cup C$ is the set whose elements are those objects which appear in at least one of A **or** B **or** C.

Example If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$, list the elements of the set $A \cup B \cup C$.

$A \cup B = \{1, 2, 3, 4, 6, 8\}$, $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$.

$B \cup C = \{2, 3, 4, 5, 6, 8\}$, $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$.

$A \cup C = \{1, 2, 3, 4, 5, 6\}$, $B \cup (A \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$.

Intersections

If A and B are sets, then $A \cap B$, read "A intersection B", is a new set. Its elements are those objects which are in A **and** in B i.e. those elements which are in both sets.

Example If $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, list the elements of the set $A \cap B$.

$\{2, 4\}$

If A and B and C are sets, their intersection $A \cap B \cap C$ is the set whose elements are those objects which appear in A **and** B **and** C i.e. those elements appearing in all three sets.

Example If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. List the elements of the set $A \cap B \cap C$.
$A \cap B = \{2, 4\}$ so $A \cap B \cap C = \{4\}$.

Universal set and complements

Given a subset A of the universal set U, the *complement* of A, denoted A^c or A', consists of all elements of U which are **not** elements in A.

Example If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$ are subsets of the universal set $U = \{1, 2, 3, \ldots, 10\}$, list the elements of the set $A^c \cup B \cup C$.

$A^c = \{5, 6, 7, 8, 9, 10\}$, $B \cup C = \{2, 3, 4, 5, 6, 8\}$,
$A^c \cup (B \cup C) = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Universal set and complements

Example Give a verbal description of the set $R \cup E \cup C$ from our class survey example. Are you in this set?

The people in the class who liked rap, classical or 80's music.

Give a verbal description of the set $(R \cup E \cup C)' = (R \cup E \cup C)^c$. Are you in this set?

The people in the class who liked none of rap, classical or 80's music.

Properties of the empty set and complements

The empty set has the following properties: For any set A,

$$\emptyset \cup A = A, \quad \emptyset \cap A = \emptyset \quad \text{and} \quad \emptyset \subset A.$$

Complements have the following properties:

$$A \cap A' = \emptyset, \quad (A')' = A \quad \text{and} \quad A \cup A' = U$$

Disjoint sets

Two sets are *disjoint* if they have no elements in common, i.e., if their intersection is the empty set.

For example if
$A = \{$All major league baseball players who hit more than 700 home runs in their career$\} = \{$ Barry Bonds, Hank Aaron, Babe Ruth$\}$ and
$B = \{$All major league baseball players with a career batting average greater than .350$\} = \{$Ty Cobb, Rogers Hornsby, Joe Jackson$\}$, then A and B are disjoint, i.e.
$A \cap B = \emptyset$.

Is there a disjoint pair from among $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 5, 7, 9\}$?

$B \cap C = \emptyset$, so B and C are disjoint.
A and B have elements in common, so they are not disjoint. Also, A and C are not disjoint.