A General Method for Constructing Planar Cognate Mechanisms

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Cognate linkages are mechanisms that share the same motion, a property that can be useful in mechanical design. This paper treats planar curve cognates, that is, planar mechanisms with rotational joints whose coupler points draw the same curve, as well as coupler cognates and timed curve cognates. The purpose of this article is to develop a straightforward method based solely on kinematic equations to construct cognates. The approach computes cognates that arise from permuting link rotations and is shown to reproduce all of the known results for cognates of four-bar and six-bar linkages. This approach is then used to construct a cognate of an eight-bar and a ten-bar linkage.

1 Introduction

Cognate linkages are mechanisms that share the same motion. Curve cognates, in particular, are distinct mechanisms, each with one degree of freedom, whose respective coupler points draw the same curve. Since cognates may occupy different regions of space and have different transmission characteristics, they can be useful in finding a more suitable mechanical design for the same function. Knowledge of cognates can also be useful when solving mechanism synthesis problems, especially in confirming that a complete solution list has been found [1, 2].

We will show how curve cognates for general planar linkages can be generated by a simple sequence of operations: form complex-vector loop equations, interchange certain link rotations, match coefficients in the kinematic equations, and then solve the resulting linear equations. An interchange of link rotations means to permute complex rotations applied to the links. For example, swapping rotations for two links means that one is aiming to construct a cognate in which the two links in the cognate linkage simply have rotational characteristics that are swapped between the corresponding two links in the original linkage. The matching of coefficients ensures that the kinematic equations hold for all input angles and thus the resulting linkage is indeed a curve cognate since it traces out the same coupler curve.

Given knowledge of which link rotations can be interchanged, the procedure is straightforward to carry out, and the results are easy to interpret graphically and ready for computer simulation. If one posits an interchange that does not correspond to a cognate, this is revealed as an inconsistency in the linear equations. The method can be used to reproduce all the known results for curve cognates of the four-bar and all the six-bars. In addition, curve cognates are also constructed for an eight-bar and a ten-bar linkage.

Even if one considers all possible permutations for the interchange of rotations, a process whose complexity grows quickly with the number of links, this does not *a priori* mean that all possible cognates have been found. Until proven otherwise, there remains the possibility that some more subtle transformation of the linkage could leave the coupler curve invariant. A companion paper [3] completes the theory of cognates for general six-bar linkages by showing that the interchange of rotations does in fact produce all possible cognates. A complete cognate theory for eight-bars and beyond is still open, but already the approach of [3] sets limits on which interchanges have a chance to produce cognates. The method presented here produces the cognates for any valid permutation one specifies.

The most famous result in cognate theory is from 1875 in which Roberts [4] showed that every four-bar coupler curve is triply generated. That result is sometimes called the

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Roberts-Chebyshev Theorem in recognition of Chebyshev's independent discovery of it three years later [5]. To our knowledge, no results on cognates of six-bar linkages were found until the work of Hartenberg and Denavit [6], followed by Roth [7] and Soni [8]. See Nolle [9] (with reference list in [10]) for a historical review as of 1974. Finally, nearly one hundred years after Roberts, Dijksman [11, 12, 13] compiled cognates for all the six-bar planar linkages. In the conference paper [14] upon which the present article expands, we showed how our method reproduces the skew pantograph, all known cognates for the four-bar, Watt six-bars, Stephenson-I and Stephenson-III six-bars, and a new eight-bar cognate. Soni also found cognates for certain eight-bars [15].

Dijksman provides the most comprehensive list of six-bar cognates by showing that cognates can be generated from permutations of link rotations and presented his results by means of intricate geometric constructions. Although correct, these drawings and their explanatory text can be rather difficult to decode thereby presenting a barrier to understanding and using those results. The purpose of this article is to present a simple method of understanding by constructing planar cognates using a complex vector approach. We are by no means the first to approach cognates from this direction; indeed, Nolle [9] states that Schor (1941), Schmid (1950), Meyer zur Capellen (1956), and Wunderlich (1958) all used some version of a complex plane formulation in treating Roberts cognates. No doubt there have been others as well since the complex plane formulation is arguably the most natural way to treat any planar linkage with all rotational joints. Rather than studying each mechanism type in isolation, our contribution is an approach that allows one to understand all the known results in terms of how a permutation of link rotations leads to a cognate and to construct new cognates for eight-bars, ten-bars, and beyond.

The rest of the paper is organized as follows. Following a list of nomenclature in Section 2, Section 3 provides background information on types of cognates and complex-vector notation. Section 4 presents our method of constructing cognates. Section 5 illustrates the method by considering in succession Roberts cognates for four-bars, the Stephenson-2A six-bar, the Watt-1A six-bar, an eight-bar, and a ten-bar. After the summary in Section 6, an appendix gives recipes for constructing all possible six-bar cognates.

2 Nomenclature

We use the following conventions and notations. Let N be the number of links and $L = N/2-1$ be the number of loops in the mechanism. One link is designated the ground and another one is designated the coupler. For example, the Watt six-bar has Watt-1 and Watt-2 inversions depending on which link is called ground, and then a choice of coupler link gives curve types Watt-1A and Watt-1B.

- The ground link is always link 0.
- Rotation θ_i, $i = 1,\dots,N-1$, is the rotation of link i relative to ground in complex-vector form, i.e., $\theta_i = e^{\sqrt{-1}\Theta_i}$, where Θ_i is the (real) rotation angle of link i.
- Link parameters a_i, b_i, c_i, $i = 0,1,\dots,N-1$, are complex vectors fixed in link i. We choose one joint of each moving link as its origin, then a_i, b_i, c_i specify the relative locations of the other joints. The origin for the ground link is chosen arbitrarily.
- θ'_i, $i = 1,\dots,N-1$, and a'_i, b'_i, c'_i, $i = 0,\dots,N-1$, are the rotations and link parameters for a cognate mechanism.
- p is the complex vector from the ground origin to the coupler point. By definition, it is the same for both the original and cognate linkages.
- Complex number γ_j, $j = 1,\dots,L$ is a non-zero scaling factor arising in the proof that loop equation j for the original mechanism is equivalent to a corresponding loop equation for its cognate.

3 Background

This section provides an introductory review to types of cognates, and to complex-vector notation as applied to planar linkages. For this paper, we restrict ourselves to planar mechanisms with rigid links connected by rotational (pin) joints. Each joint connects two links, thereby imposing one vector constraint, equivalent to two scalar constraints, requiring that the respective center points of the joint on the two links must coincide. For the purpose of classifying mechanisms with one degree of freedom, we consider only unexceptional mechanisms, being those whose number of freedoms does not change when the link dimensions are perturbed in a general fashion. For N links and J joints, these mechanisms obey the Grashof mobility criterion: $M = 3(N-1)-2J$. For $M = 1$ degree of freedom, this implies the mechanism must have $N = 2L+2$ links and $J = 3L+1$ joints, where L is the number of independent loops in the mechanism.

3.1 Types of cognates

The cognates under consideration here are curve cognates, that is, linkages which draw the same curve. We only treat cognates of the same curve type, ignoring the possibility that a six-bar might duplicate a four-bar curve or that two types of six-bars might draw the same curve.

Some curve cognates satisfy additional criteria that define subclasses of interest. A *coupler cognate* is a curve cognate where the coupler link maintains the same orientation as the original. Alternatively, after selecting an input link, a *timed curve cognate* is a curve cognate with the same functional relationship between the input rotation and the point on the coupler curve. Finally, a *timed coupler cognate* is both a coupler cognate and a timed curve cognate with respect to an input link. Once a curve cognate is found, it is straightforward to check these additional criteria and we will do so.

Another class of cognates that have been considered elsewhere are *function cognates*. These cognates maintain the functional relationship between an input crank and an output link. Some function cognates are not curve cognates, but those that are can be easily recognized here.

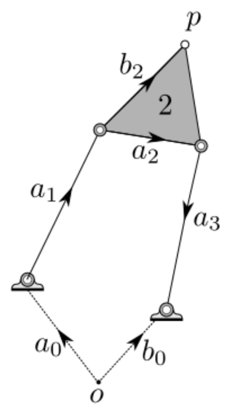

Fig. 1. Four-bar linkage

3.2 Complex-vector notation

To simplify the mathematical formulas used to represent linkages and compute cognates, we use a complex-vector formulation. Thus, a vector $[a\ b]$ in the plane is represented by a complex number $a+bi$ where $i=\sqrt{-1}$. Any complex number can be cast in the form $se^{i\Theta}$ where s is a scalar and Θ is an angle in radians. Complex arithmetic facilitates geometric transformations. In particular, complex addition implements translation, while multiplication by $se^{i\Theta}$ corresponds to a *stretch-rotation*, which stretches by s and performs a complex rotation by angle Θ. Throughout this paper, we use θ to abbreviate the complex rotation, $\theta = e^{i\Theta}$, and more specifically, after numbering the links of a mechanism, θ_j is the complex rotation of link j. By convention, the ground link, which does not move, is always link 0.

To illustrate the complex-vector notation, we begin with the case of a four-bar linkage. Referring to Figure 1, we have a loop closure equation

$$f_1 = a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3 = 0 \tag{1}$$

and a coupler-point equation

$$f_2 = a_0 + a_1\theta_1 + b_2\theta_2 = p. \tag{2}$$

Note that by subtracting one from the other, we have an alternate coupler-point equation

$$p = b_0 + (b_2 - a_2)\theta_2 - a_3\theta_3. \tag{3}$$

This is just the sum of vectors going on a different path from the origin to the coupler point. Although equivalent, one of Eq. (2) or Eq. (3) will prove more convenient depending on which cognate of the initial linkage one wishes to pursue.

The link dimensions and the placement of the ground pivots in the plane are given by a_0, b_0, a_1, a_2, b_2, a_3. To compute the mechanism's motion, consider that given one link rotation, say θ_1, one can solve Eq. (1) for θ_2 and θ_3, keeping in mind that the complex loop equation is equivalent to two scalar equations (by taking its real and imaginary parts) and the rotations are each parameterized by a single scalar angle. Then, one can evaluate the coupler point position using Eq. (2) or Eq. (3). A more facile approach based on using the complex conjugate of the loop equation is presented in [16, 17]. This paper does not need to solve the loop and coupler-point equations; instead, we merely need to show that cognate mechanisms satisfy the same equations, i.e., trace the same coupler curve as the original mechanism.

4 Cognate Construction Procedure

The key steps in our method for constructing curve cognates for an L-loop mechanism are as follows. It can be seen that including the origin point in the ground link and the coupler point in the coupler link, an unexceptional, mobility-1, L-loop mechanism has $4L+2$ independent link parameters and $N=2L+2$ links. Let $q=(q_1,\ldots,q_{4L+2})$ be the link parameters of the original linkage and q' be those for the cognate. Similarly, let $\theta=(\theta_1,\ldots,\theta_{2L+1})$ be the link rotations for the original linkage and θ' the rotations for the cognate.

1. Choose a permutation P for interchanging link rotations between the original and the cognate mechanism. Hence, we are seeking a cognate whose link rotations are $\theta' = P(\theta)$.
2. For the original mechanism, form L independent complex-vector loop equations,
$$f_1(q,\theta)=0,\ldots,f_L(q,\theta)=0$$
and one coupler-point equation for a complex-vector path from the origin to the coupler point, p,
$$p = f_{L+1}(q,\theta).$$
3. Similarly, for the cognate mechanism, write equations
$$f_1(q',\theta')=0,\ldots,f_L(q',\theta')=0,\ f_{L+1}(q',\theta')=p.$$
4. Substitute $\theta'=P(\theta)$ into equations from Step 3 yielding
$$f_1'(q',P(\theta))=0,\ldots,f_L'(q',P(\theta))=0,\ f_{L+1}'(q',P(\theta))=p.$$
5. If the corresponding equations in Steps 3 and 4 do not contain the same set of link rotations, replace equations
$$f_1(q,\theta)=0,\ldots,f_L(q,\theta)=0, f_{L+1}(q,\theta)=p$$
with independent linear combinations to allow every link rotation to be properly matched in the final step.
6. Set each loop function for the cognate equal to a stretch-rotation of its corresponding function from the original:
$$f_i'(q',P(\theta)) = \gamma_i f_i(q,\theta), \quad i=1,\ldots,L.$$
7. Set the coupler points equal:
$$f_{L+1}'(q',P(\theta)) = f_{L+1}(q,\theta).$$
8. Solve for the cognate parameters q' and stretch-rotations $\gamma_1,\ldots,\gamma_L$ by matching the coefficients of the link rotations and constant term in the loop (Step 6) and coupler-point (Step 7) equations.

The final solution is a mechanism that satisfies the same set of loop equations and the same coupler-point equation as the original mechanism, hence it is a curve cognate. If the permutation chosen at Step 1 does not alter the rotation of the input link, then we obtain a timed curve cognate. If the permutation does not alter the rotation of the coupler link, then we obtain a coupler cognate. It is sometimes possible to satisfy both of these and thus obtain a timed coupler cognate.

For the matching procedure to succeed in Step 8, the equations produced in Steps 6-7 must have the same link rotations appearing on both sides. For example, if link rotation 1 is to be interchanged with link rotation 2, then every equation must involve either both links 1 and 2 or neither of them. Usually, but not always, with appropriate choices of loops and paths in Steps 2-3, one is able to match rotations without needing rearrangement in Step 5.

If rotations match properly, the process of solving for the cognate in Step 8 is simple. As illustrated in Eqs. (1-3), the equations in the complex-vector formulation are linear in the parameters and the only unknowns on the right-hand side of the equations in Steps 6-7 are the stretch-rotations, which also appear linearly. Hence, the equations to solve in Step 8 are all linear and so the system is straightforward to solve symbolically. We note that not all permutations one might consider at Step 1 lead to cognates: invalid choices become apparent in Step 8 as an inconsistent set of linear equations. This may happen if the number of coefficients to match exceeds the number of unknowns. On the other hand, it may also happen that the number of coefficients to match is smaller than the number of unknowns, in which case there exists a positive-dimensional set of cognates. In particular, this happens for the Watt-1A mechanism, which has a two-dimensional family of curve cognates (see Section 5.3).

5 Examples

Our procedure becomes much clearer when applied to a specific mechanism. We will first illustrate it using the simplest example of interest, the four-bar linkage, followed by a Stephenson-2A six-bar, the Watt-1A six-bar, an eight-bar, and a ten-bar. An appendix summarizes how to derive cognates for all the planar six-bar curve mechanisms.

5.1 Four-bar: Roberts cognates

The existence of three curve cognates for every four-bar linkage was first proved by Roberts [4]. Figure 2 shows their geometric construction, as was found by Chebyshev [5] and Cayley [18]. This arrangement contains three similar triangles, links 2, $2'$, and $2''$, and three parallelograms. A proof that the four-bars labeled "swap 1-2" and "swap 2-3" are truly cognates of the original requires showing that point c_0 stays fixed as the original four-bar moves. In a geometric approach, this fact follows from showing that the focal triangle a_0, b_0, c_0 is also similar to the coupler triangle, link 2. Our approach proves this while at the same time generating the formulas for the cognate link parameters and for the location of point c_0.

In the Chebyshev-Cayley construction shown in Fig. 2, the parallelogram attached to a_0 shows that link $1'$ has the same rotation link 2 and link $2'$ has the same rotation as link 1. That is, the cognate linkage labeled "swap 1-2" interchanges the rotations of links 1 and 2. In addition, the other two parallelograms imply that link $3'$ and link 3 undergo the same rotation. Hence the "swap 1-2" cognate has rotations $(\theta'_1, \theta'_2, \theta'_3) = (\theta_2, \theta_1, \theta_3)$. As implied by its label, the "swap 2-3" cognate obeys a different permutation of the link rotations, namely $(\theta''_1, \theta''_2, \theta''_3) = (\theta_1, \theta_3, \theta_2)$. The key observation Dijksman used in finding six-bar cognates was that these also involve permuting link rotations. Our methodology derives from this observation: once a valid permutation of the link rotations is specified, all the cognate linkage parameters are determined by solving a system of linear equations.

We begin by considering the "swap 1-2" cognate. The four-bar has 1 loop and 4 links, and its 6 link parameters are denoted $a_0, b_0, a_1, a_2, b_2, a_3$. In Step 1, we consider the permutation $(\theta'_1, \theta'_2, \theta'_3) = (\theta_2, \theta_1, \theta_3)$. The loop equation Eq. (1) and the coupler-point equation Eq. (2) are used for the original mechanism in Step 2 and for the cognate mechanism in Step 3 with a'_0 in place of a_0, θ'_1 in place of θ_1, and so on. After the substitution in Step 4, Step 5 is unnecessary since the same set of link rotations appear in f_1, f'_1 and in f_2, f'_2, respectively. Steps 6-7 yield

$$f'_1 = \gamma_1 f_1 : \quad a'_0 - b'_0 + a'_1\theta_2 + a'_2\theta_1 + a'_3\theta_3 = \gamma_1(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3), \tag{4}$$

$$f'_2 = f_2 : \quad a'_0 + a'_1\theta_2 + b'_2\theta_1 = a_0 + a_1\theta_1 + b_2\theta_2. \tag{5}$$

In Step 8, equating the coefficients of the link rotations 1, θ_1, θ_2, θ_3 on both sides of these equations yields a set of seven linear equations in seven unknown parameters, these being $(a'_0, b'_0, a'_1, a'_2, b'_2, a'_3)$ for the cognate linkage and the stretch-rotation γ_1. Listing these out, the loop equation Eq. (4) gives

$$a'_0 - b'_0 = \gamma_1(a_0 - b_0),\ a'_2 = \gamma_1 a_1,\ a'_1 = \gamma_1 a_2,\ a'_3 = \gamma_1 a_3 \tag{6}$$

while the coupler-point equation Eq. (5) yields

$$a'_0 = a_0, \qquad b'_2 = a_1, \qquad a'_1 = b_2. \tag{7}$$

Since $a'_1 = \gamma_1 a_2$ and $a'_1 = b_2$, one finds that $\gamma_1 = b_2/a_2$, which is the stretch-rotation that transforms a_2 into b_2. With γ_1 known, all of the cognate parameters are easily determined from Eqs. (6,7). In particular, one sees that ground pivot a'_0 stays fixed, i.e., $a'_0 = a_0$, whereas ground pivot b'_0 moves to a new location $b'_0 = c_0 := a_0 + (b_2/a_2)(b_0 - a_0)$. We label this new ground pivot as c_0 in Figure 2. As is well-known, (a_0, b_0, c_0) are the singular foci of the four-bar's coupler curve and they form a triangle that is similar to the coupler triangle, $(0, a_2, b_2)$ [19].

The simple linear relations of Eqs. (6,7) directly imply the parallelograms and similar triangles in the Chebyshev-Cayley geometric construction. In particular, $b'_2 = a_1$, $a'_1 = b_2$ in Eq. (7) imply that quadrilateral a_1, b_2, b'_2, a'_1 is a parallelogram, while $b'_2 = a_1, a'_2 = \gamma_1 a_1$, hence $\gamma_1 = a'_2/b'_2$,

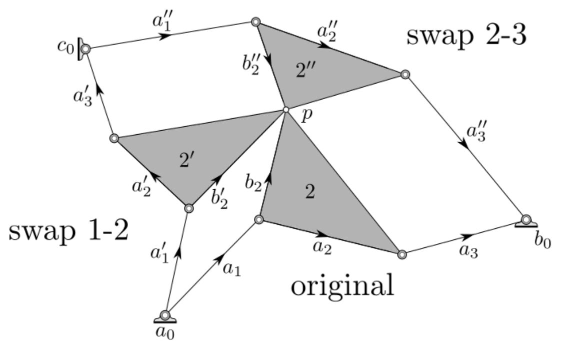

Fig. 2. Roberts cognates with numerical values of the parameters presented in Example 5.1

along with $\gamma_1 = b_2/a_2$ shows that triangle $2'$ is similar to triangle 2. These geometric relations automatically appear using our approach.

A second cognate is found by swapping rotations between links 2 and 3. Call its parameters $a''_0, \ldots, a''_3$ to distinguish them from those for the first cognate. The same procedure applies, but to successfully match terms in Step 8, we can use the alternate coupler-point equation Eq. (3) at Step 3. Since Eq. (3) was obtained by subtracting Eq. (1) from Eq. (2), this preliminary rearrangement avoids needing any additional rearrangement in Step 5. This time, one finds that $b''_0 = b_0$ stays in place while a_0 moves to the third singular focus, $a''_0 = b'_0 = c_0$. Matching all coefficients we find the stretch rotation factor, call it ζ_1, to be

$$\zeta_1 = 1 - \frac{b_2}{a_2}$$

with cognate link parameters

$$\begin{aligned} a''_0 &= c_0 & b''_0 &= b_0 & a''_1 &= \zeta_1 a_1 \\ a''_2 &= \zeta_1 a_3 & b''_2 &= a_3(\zeta_1 - 1) & a''_3 &= \zeta_1 a_2. \end{aligned}$$

One may also swap rotations between links 1 and 3. To carry through the matching procedure without needing Step 5, one picks Eq. (2) at Step 2 and Eq. (3) at Step 3. Thus, after the substitution $\theta'_1 = \theta_3$ and $\theta'_3 = \theta_1$ at Step 4, the same rotations, namely θ_1 and θ_2 appear on both sides in Step 7. The result computed in Step 8 is valid but it does not produce a new four-bar. Instead, this simply produces the original four-bar with its links renumbered 0-3-2-1 in place of 0-1-2-3. While this renumbering is not interesting from a mechanical standpoint, it is meaningful algebraically. As shown in [2], the system of path-synthesis equations for a four-bar coupler curve to pass through nine given precision points has 8652 solutions that arise in a six-way symmetry: three cognates which each allow a two-way renumbering. Hence, the 8652 solutions correspond with 1442 distinct coupler curves, each of which is generated by three four-bar curve cognates.

Finally, we observe that the first cognate $(a'_0, \ldots, a'_3)$ is a timed curve cognate if link 3 is the input, while the second cognate $(a''_0, \ldots, a''_3)$ is if link 1 is the input. Neither is a coupler cognate since they do not preserve the rotation of link 2.

Example 5.1. The table below lists the parameters for the four-bar linkage along with the parameters (to 4 decimal places) for the cognates derived above and drawn in Figure 2.

	Original	Swap 1-2	Swap 2-3
a_0	$0.0 + 0.0i$	$0.0000 + 0.0000i$	$-0.6549 + 2.2196i$
b_0	$3.0 + 0.8i$	$-0.6549 + 2.2196i$	$3.0000 + 0.8000i$
a_1	$0.8 + 0.8i$	$0.2000 + 0.9000i$	$1.4118 + 0.2196i$
a_2	$1.2 - 0.3i$	$-0.6118 + 0.5804i$	$1.2431 - 0.4392i$
b_2	$0.2 + 0.9i$	$0.8000 + 0.8000i$	$0.2431 - 0.7392i$
a_3	$1.0 + 0.3i$	$-0.2431 + 0.7392i$	$1.0000 - 1.2000i$

5.2 Stephenson-2A

The Stephenson-2 six-bar has two coupler curve types depending on which link is designated the coupler. Figure 3 shows the Stephenson-2A option where link 5 carries the coupler point. (The alternative, Stephenson-2B, places the coupler point on link 2.) Cognates for the Stephenson-2A appear as a group of four: the original, swap rotations 2-3, swap rotations 4-5, and swap both 2-3 and 4-5. We will show how to derive all the Stephenson-2A curve cognates and also illustrate what goes wrong for an inadmissible permutation.

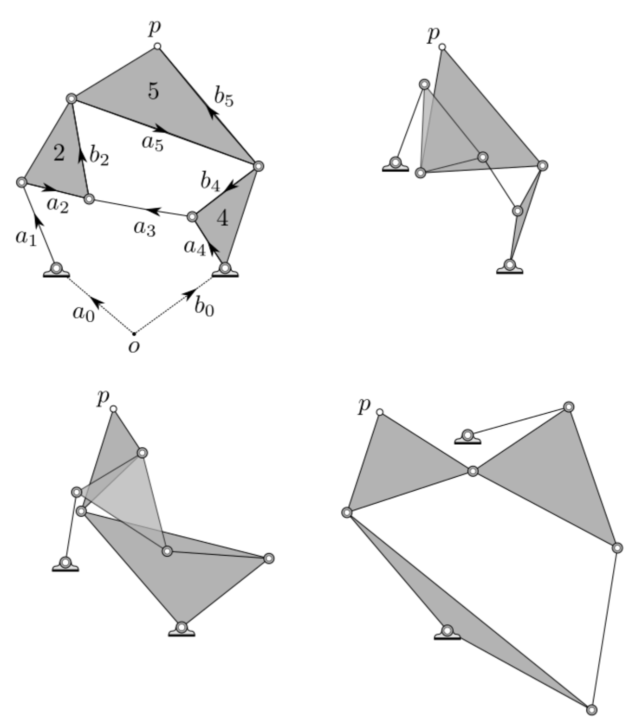

Fig. 3. Stephenson-2A mechanism (top left), swap 2-3 (top right), swap 4-5 (bottom left), and swap 2-3 and 4-5 (bottom right) with numerical values of the parameters presented in Example 5.2

5.2.1 Swap rotations 2-3

To derive the cognate that swaps link rotations 2 and 3, we wish to form loop and coupler-point equations that either contain both or neither of links 2 and 3. Loops 0-1-2-3-4 and 2-3-4-5 suffice, as does the path to the coupler point using links 0-4-5. Accordingly, at Step 2, we have

$$f_1 = a_0 - b_0 + a_1\theta_1 + a_2\theta_2 - a_3\theta_3 - a_4\theta_4 = 0, \quad (8)$$
$$f_2 = b_2\theta_2 + a_3\theta_3 + b_4\theta_4 + a_5\theta_5 = 0, \quad (9)$$
$$f_3 = b_0 + (a_4 - b_4)\theta_4 + b_5\theta_5 = p. \quad (10)$$

At Step 3, we use the same equations with a'_0 in place of a_0, θ'_1 in place of θ_1, and so on. After substituting $\theta'_2 = \theta_3$, and $\theta'_3 = \theta_2$, Steps 6-7 give

$$\begin{aligned}
f'_1 =& \gamma_1 f_1 : \\
& a'_0 - b'_0 + a'_1\theta_1 + a'_2\theta_3 - a'_3\theta_2 - a'_4\theta_4 \\
& \qquad = \gamma_1(a_0 - b_0 + a_1\theta_1 + a_2\theta_2 - a_3\theta_3 - a_4\theta_4), \\
f'_2 =& \gamma_2 f_2 : \\
& b'_2\theta_3 + a'_3\theta_2 + b'_4\theta_4 + a'_5\theta_5 \\
& \qquad = \gamma_2(b_2\theta_2 + a_3\theta_3 + b_4\theta_4 + a_5\theta_5), \\
f'_3 =& f_3 : \\
& b'_0 + (a'_4 - b'_4)\theta_4 + b'_5\theta_5 \\
& \qquad = b_0 + (a_4 - b_4)\theta_4 + b_5\theta_5.
\end{aligned}$$

Matching terms on the left to those on the right of these equalities, one obtains 12 linear conditions in the 12 unknowns, these being the 10 cognate link parameters, $a'_0, \ldots, b'_5$, and the two stretch-rotations γ_1, γ_2. The 12 conditions are quite simple and sparse, including

$$\begin{aligned}
& a'_0 - b'_0 = \gamma_1(a_0 - b_0), \quad a'_1 = \gamma_1 a_1, \quad a'_2 = -\gamma_1 a_3, \\
& \qquad \ldots, \quad b'_0 = b_0, \quad a'_4 - b'_4 = a_4 - b_4, \quad b'_5 = b_5. \quad (11)
\end{aligned}$$

From the whole set of 12 equations, one finds the stretch-rotations to be

$$\gamma_1 = \frac{b_2(a_4 - b_4)}{b_2 a_4 + a_2 b_4}, \quad \gamma_2 = \frac{a_2(b_4 - a_4)}{b_2 a_4 + a_2 b_4},$$

and the cognate link parameters as

$$\begin{array}{lll}
 & a'_0 = b_0 + \gamma_1(a_0 - b_0) & \\
b'_0 = b_0 & a'_1 = \gamma_1 a_1 & a'_2 = -\gamma_1 a_3 \\
b'_2 = \gamma_2 a_3 & a'_3 = -\gamma_1 a_2 & a'_4 = \gamma_1 a_4 \\
b'_4 = \gamma_2 b_4 & a'_5 = \gamma_2 a_5 & b'_5 = b_5.
\end{array}$$

Links 1 and 4 adjacent to ground keep their original rotations as does the coupler link 5. Accordingly, this cognate is a timed coupler cognate for input at either link 1 or 4.

5.2.2 Swap rotations 4-5

We will not write out the details for deriving the cognate obtained by swapping the rotations of links 4 and 5. One may find it similarly to the procedure in section 5.2.1 by using loops 0-1-2-5-4 and 2-3-4-5, and the path 0-4-5 to the coupler point. Since links 4 and 5 appear in every equation, these satisfy the matching condition so Step 5 is not utilized. Moreover, we again obtain 12 linear conditions in 12 unknowns which are readily solved for the required stretch-rotations,

$$\gamma_1 = \frac{-b_5}{a_5}, \quad \gamma_2 = \frac{b_5(a_4 - b_4)}{a_5 b_4},$$

and the cognate mechanism's parameters,

$$\begin{array}{lll}
 & a'_0 = b_0 + \gamma_1(a_0 - b_0) & \\
b'_0 = b_0 & a'_1 = \gamma_1 a_1 & a'_2 = \gamma_1(a_2 + b_2) - \gamma_2 b_2 \\
b'_2 = \gamma_2 b_2 & a'_3 = \gamma_2 a_3 & a'_4 = a_5(\gamma_2 - \gamma_1) \\
b'_4 = \gamma_2 a_5 & a'_5 = \gamma_2 b_4 & b'_5 = a_4 - b_4.
\end{array}$$

Since link 1 keeps its original rotation, this cognate is a timed curve cognate for input at link 1.

5.2.3 Swap both rotations 2-3 and 4-5

Once one has the formulas in hand for swapping 2-3 and for swapping 4-5, one can compute the result of swapping both by applying the two sets of formulas in sequence. The order of the sequence, 2-3 then 4-5 versus 4-5 then 2-3, doesn't matter: the same final cognate results.

Even though the sequential option is available, let's consider how to derive the cognate as one double-swap. This will illustrate a case where Step 5 comes into play. The trouble is that although the loop 2-3-4-5 and the coupler path 0-4-5 both satisfy the matching criterion, there is no second loop equation that does so. The possibilities for a second loop are 0-1-2-3-4 which swaps to 0-1-3-2-5 as well as the loop 0-1-2-5-4 which swaps to 0-1-3-4-5. For both, we see that the cognate loop has a link rotation that is not matched in the original. The way out of this bind is to use a linear combination of the loops to eliminate the unmatched rotation.

We already wrote the equations for loops 0-1-2-3-4 and 2-3-4-5 and for path 0-4-5 to the coupler point as f_1, f_2, f_3 in Eqs. (8,9,10). After the double-swap, we get rotations 0-1-3-2-5 in f'_1. To match these, consider the linear combination that eliminates θ_4 between f_1 and f_2:

$$\begin{aligned}
g_1 :=& b_4 f_1 + a_4 f_2 \\
=& b_4(a_0 - b_0) + b_4 a_1\theta_1 + (b_4 a_2 + a_4 b_2)\theta_2 \quad (12) \\
& \qquad + a_3(a_4 - b_4)\theta_3 + a_4 a_5\theta_5 = 0.
\end{aligned}$$

This combination contains rotations for links 0-1-2-3-5, which are the same ones that appear in loop $0'$-$1'$-$2'$-$3'$-$4'$ after the double-swap of rotations turns it into 0-1-3-2-5.

Therefore, at Step 5, f_1 is replaced by g_1 yielding at Steps 6-7 the following equations:

$$f_1'(q',P(\theta)) = \gamma_1 g_1(q,\theta), \qquad f_2'(q',P(\theta)) = \gamma_2 f_2(q,\theta),$$
$$f_3'(q',P(\theta)) = f_3(q,\theta)$$

where $P(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = (\theta_1,\theta_3,\theta_2,\theta_5,\theta_4)$.

After matching coefficients we again end up with 12 linear equations in 12 unknowns and find the stretch-rotation factors to be

$$\gamma_1 = \frac{b_2 b_5}{a_2 a_5}, \quad \gamma_2 = -\frac{b_5(a_2 b_4 + a_4 b_2)}{a_2 b_4 a_5}$$

and the cognate parameters as

$$\begin{array}{lll} & a_0' = b_0 + \gamma_1(a_0 - b_0) & \\ b_0' = b_0 & a_1' = \gamma_1 a_1 & a_2' = \gamma_1 a_3(\frac{a_4}{b_4} - 1) \\ b_2' = \gamma_2 a_3 & a_3' = \gamma_2 b_2 & a_4' = b_5 + \gamma_2 a_5 \\ b_4' = \gamma_2 a_5 & a_5' = \gamma_2 b_4 & b_5' = a_4 - b_4. \end{array}$$

Example 5.2. The table below lists the parameters for the Stephenson-2A six-bar mechanism along with the parameters (to 4 decimal places) for the three cognates derived above and drawn in Figure 3.

	Original	Swap 2-3	Swap 4-5	Swap 2-3 & 4-5
a_0	$0.0+0.0i$	$0.3204+0.6180i$	$0.3139+0.3869i$	$1.1198+1.1559i$
b_0	$1.0+0.0i$	$1.0000+0.0000i$	$1.0000+0.0000i$	$1.0000+0.0000i$
a_1	$-0.2+0.5i$	$0.1731+0.4634i$	$0.0562+0.4204i$	$0.6019+0.1713i$
a_2	$0.4-0.1i$	$0.3460-0.4388i$	$0.5535-0.3591i$	$0.2957-0.8438i$
b_2	$-0.1+0.6i$	$-0.3662-0.0965i$	$-0.1542+0.5861i$	$-0.8562+0.4630i$
a_3	$-0.6+0.1i$	$-0.2100+0.3152i$	$-0.6044+0.0442i$	$0.1603+0.9601i$
a_4	$-0.2+0.3i$	$0.0495+0.3275i$	$0.5280+0.4040i$	$0.8572-0.4767i$
b_4	$-0.4-0.3i$	$-0.1505-0.2725i$	$1.1280-0.2960i$	$1.4572-1.1767i$
a_5	$1.1-0.4i$	$0.7268+0.0538i$	$-0.3693-0.3343i$	$-0.7612-0.2463i$
b_5	$-0.6+0.7i$	$-0.6000+0.7000i$	$0.2000+0.6000i$	$0.2000+0.6000i$

5.2.4 An inadmissible permutation: swap rotations 2-5

One may wonder why other permutations besides the swapping of 2-3 and 4-5 do not give curve cognates for the Stephenson-2A. To illustrate, consider swapping the rotations of links 2 and 5. Loops 0-1-2-5-4 and 2-3-4-5 and vector path 0-1-2-5 all contain both 2 and 5, so the matching condition is satisfied. The trouble is that in contrast to the swaps previously considered, this time the path to the coupler point traverses four instead of just three links. The consequence is that we have 13 coefficients to match, and we only have 12 unknowns. A row reduction procedure shows that these equations are in general incompatible, so rotations 2 and 5 cannot be interchanged.

5.3 Watt-1A six-bar

Among the six-bar coupler curve mechanisms, only the Watt-1A, shown in Figure 4, has a positive-dimensional set

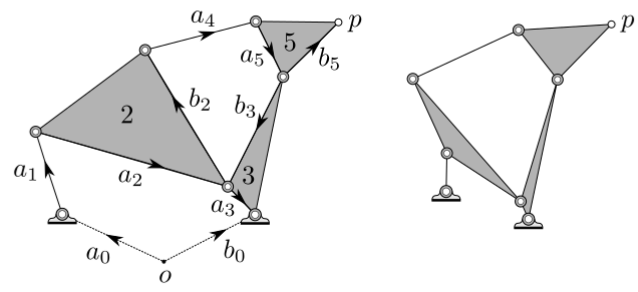

Fig. 4. Watt-1A mechanism (left) and a cognate (right) with numerical values of the parameters presented in Example 5.3

of curve cognates. Let's see how our method of constructing cognates deals with this.

The Watt-1A cognates derive from the trivial permutation: no swaps at all. The loops are 0-1-2-3 and 2-3-5-4 and a path to the coupler point is 0-3-5. The number of coefficients to match in these equations is $4+4+3=11$, and as is always the case for six-bars, we have 12 unknowns: 10 link parameters and 2 stretch-rotations. Accordingly, we can specify one link vector, say a_0', and still satisfy all the conditions imposed by matching coefficients of the rotations. Since a link vector has a real and an imaginary part, the set of curve cognates is two-real-dimensional.

Specifying a_0', we find the stretch-rotation factors to be

$$\gamma_1 = \frac{a_0' - b_0}{a_0 - b_0}, \quad \gamma_2 = 1 + \frac{a_3(a_0 - a_0')}{b_3(a_0 - b_0)}$$

and the cognate mechanism link parameters

$$\begin{array}{lll} b_0' = b_0 & a_1' = \gamma_1 a_1 & a_2' = \gamma_1 a_2 \\ b_2' = \gamma_2 b_2 & a_3' = \gamma_1 a_3 & b_3' = \gamma_2 b_3 \\ a_4' = \gamma_2 a_4 & a_5' = \gamma_2 a_5 & b_5' = b_5. \end{array}$$

Since all the links of the cognate keep their original rotations, the result is a timed coupler cognate.

Example 5.3. The table below lists the parameters for the Watt-1A six-bar linkage along with the parameters (to 4 decimal places) for a cognate with $a_0' = 0.4+0.1i$. Both mechanisms are drawn in Figure 4.

	Original	Cognate		Original	Cognate
a_0	$0.0+0.0i$	$0.4000+0.1000i$	a_3	$0.1-0.1i$	$0.0286-0.0571i$
b_0	$0.7+0.0i$	$0.7000+0.0000i$	b_3	$-0.2-0.4i$	$-0.1286-0.4429i$
a_1	$-0.1+0.3i$	$0.0000+0.1429i$	a_4	$0.4+0.1i$	$0.3871+0.1757i$
a_2	$0.7-0.2i$	$0.2714-0.1857i$	a_5	$0.1-0.2i$	$0.1386-0.1843i$
b_2	$-0.3+0.5i$	$-0.3971+0.4514i$	b_5	$0.2+0.2i$	$0.2000+0.2000i$

5.4 An eight-bar cognate

So far, we have used our approach to provide simple demonstrations of known results in cognate theory. However, the beauty of the approach is that it easily generates cognates for more complex linkages. To show this, we generate a novel cognate of the eight-bar linkage in Figure 5.

We can generate a cognate by swapping the rotations of links 1 and 2. Three compatible loops are 0-1-2-3, 1-2-3-5-4, and 4-5-7-6. The vector path 0-3-5-7 to the coupler point is also compatible. There are $4+5+4+4 = 17$ coefficients to match. The number of link parameters is $4L+2 = 14$ and there will be 3 stretch-rotations associated to matching the loops for a total of 17 unknowns. Therefore, we can expect that this swap will lead to a unique cognate.

The relevant loop and coupler-point equations are:

$$0 = a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_3\theta_3, \tag{13}$$
$$0 = b_1\theta_1 - a_2\theta_2 - b_3\theta_3 + a_4\theta_4 - b_5\theta_5, \tag{14}$$
$$0 = b_4\theta_4 - a_5\theta_5 + a_6\theta_6 + a_7\theta_7, \tag{15}$$
$$p = b_0 + (b_3 - a_3)\theta_3 + (a_5 + b_5)\theta_5 + b_7\theta_7. \tag{16}$$

The cognate uses these same equations except interchanging rotations 1 and 2. After introducing three stretch-rotation factors, one for each loop equation Eqs. (13)-(15), and equating coefficients, one obtains stretch-rotations

$$\gamma_1 = \frac{b_1(a_3 - b_3)}{a_1 b_3 + b_1 a_3}, \qquad \gamma_2 = \frac{a_1(b_3 - a_3)}{a_1 b_3 + b_1 a_3},$$
$$\gamma_3 = \frac{a_1(a_3 b_5 + b_3 a_5) + b_1 a_3 (a_5 + b_5)}{a_5(a_1 b_3 + a_3 b_1)},$$

and cognate mechanism link parameters

$$\begin{aligned}
&a_0' = \gamma_1 a_0 + (1-\gamma_1) b_0, & & & b_0' &= b_0,\\
&a_1' = \gamma_1 a_2, & b_1' &= -\gamma_2 a_2, & a_2' &= \gamma_1 a_1,\\
&a_3' = \gamma_1 a_3, & b_3' &= \gamma_2 b_3, & a_4' &= \gamma_2 a_4,\\
&b_4' = \gamma_3 b_4, & a_5' &= \gamma_3 a_5, & b_5' &= \gamma_2 b_5,\\
&a_6' = \gamma_3 a_6, & a_7' &= \gamma_3 a_7, & b_7' &= b_7.
\end{aligned}$$

Since rotation 7 is preserved, this is a coupler cognate. When link 3 is the input crank, this is a timed coupler cognate. If instead link 1 is the input, timing is not preserved.

Example 5.4. The table below lists the parameters for the eight-bar linkage along with the parameters (to 4 decimal places) for the cognate derived above and drawn in Figure 5.

	Original	Cognate		Original	Cognate
a_0	$-4.0+0.0i$	$-2.2665+1.2640i$	a_4	$2.1+0.5i$	$0.7494+1.4184i$
b_0	$0.0+0.0i$	$0.0000+0.0000i$	b_4	$-1.5+0.4i$	$-1.4238-0.6638i$
a_1	$0.8+1.0i$	$1.4797-0.6767i$	a_5	$2.5+0.6i$	$1.5503+2.0892i$
b_1	$-1.0+0.5i$	$-1.1130-1.4949i$	b_5	$-2.3+0.4i$	$-1.3503-1.0892i$
a_2	$2.5+0.2i$	$0.7693+0.3138i$	a_6	$1.9+0.6i$	$1.0847+1.6995i$
a_3	$0.7-1.2i$	$0.0174-0.9011i$	a_7	$2.1-0.4i$	$1.8894+1.0534i$
b_3	$0.9+0.4i$	$0.2174+0.6989i$	b_7	$-0.5+0.9i$	$-0.5000+0.9000i$

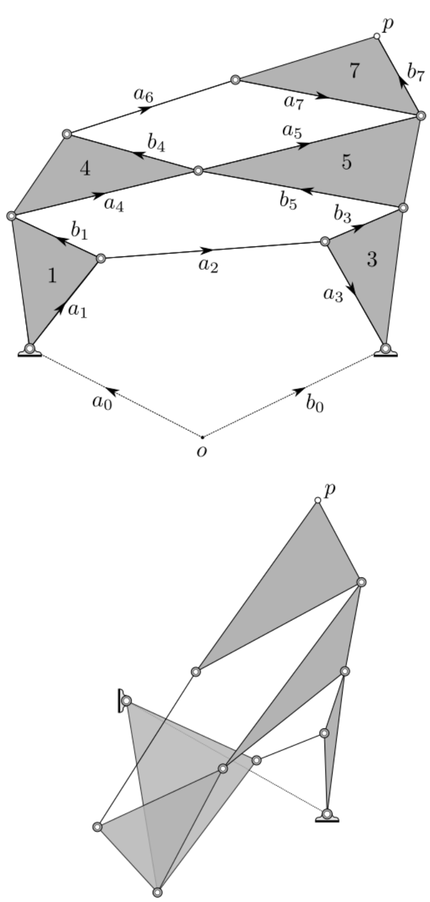

Fig. 5. An eight-bar mechanism (top) and a cognate (bottom) with numerical values of the parameters presented in Example 5.4

5.5 A ten-bar cognate

As a final demonstration of the simplicity of the approach, we use the method to generate a novel cognate to the ten-bar linkage in [20, Fig. 15] that is shown in Figure 6.

The following loop and coupler-point equations are compatible with swapping link rotations 3 and 4:

$$0 = a_0 - b_0 + a_1\theta_1 + a_2\theta_2 + a_6\theta_6, \tag{17}$$
$$0 = b_0 - c_0 + b_2\theta_2 + a_3\theta_3 + a_4\theta_4, \tag{18}$$
$$0 = b_3\theta_3 + b_4\theta_4 + b_5\theta_5 + a_8\theta_8, \tag{19}$$
$$\begin{aligned}0 = a_0 - c_0 + (a_1 - b_1)\theta_1 + a_3\theta_3 \\ - b_4\theta_4 - a_5\theta_5 - a_7\theta_7 - a_9\theta_9,\end{aligned} \tag{20}$$
$$p = a_0 + (a_1 - b_1)\theta_1 - a_7\theta_7 + b_9\theta_9. \tag{21}$$

A count of the conditions to be satisfied from matching coefficients is $4+4+4+7+4 = 23$ while there are just 22 unknowns: 18 link parameters and 4 stretch-rotation factors. By that count, one might hastily conclude that there are too few freedoms to match all the conditions, but it turns out that the conditions are in fact compatible. In particular, since neither θ_1 nor θ_7 is involved in the swap, their coefficients in

Eqs. (20,21) lead to the following four equations:

$$a_1' - b_1' = \gamma_4(a_1 - b_1), \qquad a_7' = \gamma_4 a_7,$$
$$a_1' - b_1' = a_1 - b_1, \qquad a_7' = a_7.$$

It is easily seen that these four equations only place three conditions on the parameters. Carrying out the entire procedure, one finds a unique solution for the stretch-rotations:

$$\gamma_1 = \frac{a_0 a_4 + b_4 b_0 - b_4 c_0 - c_0 a_4}{a_4(a_0 - b_0)}, \qquad \gamma_2 = -\frac{b_4}{a_4},$$
$$\gamma_3 = -\frac{a_3}{b_3}, \qquad \gamma_4 = 1,$$

and the link parameters:

$$\begin{array}{lll} a_0' = a_0 & b_0' = c_0 + \gamma_2(b_0 - c_0) & c_0' = c_0 \\ a_1' = \gamma_1 a_1 & b_1' = b_1 + a_1(\gamma_1 - 1) & a_2' = \gamma_1 a_2 \\ b_2' = \gamma_2 b_2 & a_3' = -b_4 & b_3' = \gamma_3 b_4 \\ a_4' = \gamma_2 a_3 & b_4' = -a_3 & a_5' = a_5 \\ b_5' = \gamma_3 b_5 & a_6' = \gamma_1 a_6 & a_7' = a_7 \\ a_8' = \gamma_3 a_8 & a_9' = a_9 & b_9' = b_9. \end{array}$$

This is a coupler cognate since the rotation of link 9 is preserved. When link 1 or 2 is the input, this is a timed coupler cognate. When link 3 is the input, timing is not preserved.

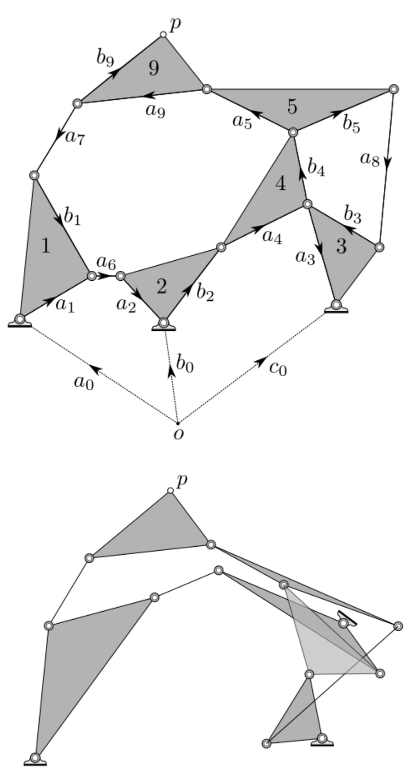

Fig. 6. A ten-bar mechanism (top) and a cognate (bottom) with numerical values of the parameters presented in Example 5.5

Example 5.5. The table below lists the parameters for the ten-bar linkage along with the parameters (to 4 decimal places) for the cognate derived above and drawn in Figure 6.

	Original	Cognate		Original	Cognate
a_0	$0.0+0.0i$	$0.0000+0.0000i$	a_4	$0.6+0.3i$	$-0.5533-0.0067i$
b_0	$1.0+0.0i$	$2.3667+1.0000i$	b_4	$-0.1+0.5i$	$-0.2000+0.7000i$
c_0	$2.2+0.1i$	$2.2000+0.1000i$	a_5	$-0.6+0.3i$	$-0.6000+0.3000i$
a_1	$0.5+0.3i$	$0.8833+1.2100i$	b_5	$0.7+0.3i$	$0.8941-0.3235i$
b_1	$0.4-0.7i$	$0.7833+0.2100i$	a_6	$0.2+0.0i$	$0.4733+0.2000i$
a_2	$0.3-0.3i$	$1.0100-0.4100i$	a_7	$-0.3-0.5i$	$-0.3000-0.5000i$
b_2	$0.4+0.5i$	$0.2867-0.3933i$	a_8	$-0.1-1.1i$	$-1.0294-0.9176i$
a_3	$0.2-0.7i$	$0.1000-0.5000i$	a_9	$-0.9-0.1i$	$-0.9000-0.1000i$
b_3	$-0.5+0.3i$	$0.3353+0.5412i$	b_9	$0.6+0.5i$	$0.6000+0.5000i$

6 Conclusion

We have presented a method of deriving planar curve cognates and illustrated its application to the four-bar, several six-bars, one eight-bar, and one ten-bar linkage. Cognates are found by interchanging link rotations in a complex-vector formulation of loop and coupler-point equations, resulting in formulas that are easy to apply, especially in a computer graphics environment. As we saw in Section 5.2.4, not all permutations lead to valid cognates. Thus, at the level of development presented here, some trial-and-error would be required to find all cognates of a given linkage type for linkages with $N > 6$. A companion paper [3] addresses the issue of finding all possible valid permutations.

Appendix A summarizes how to derive all known six-bar curve cognates. Beyond giving simple derivations for known cognates, the procedure also allows one to produce new cognates as demonstrated in Sections 5.4 and 5.5 by finding a novel cognate of an eight-bar and ten-bar linkage.

Acknowledgements

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A Deriving all six-bar cognates

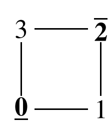
For easy reference, we provide a quick summary of how to derive all planar four-bar and six-bar cognates. For each curve type, we provide:

- the mechanism's type graph (which represents links as nodes and joints as edges), where the ground link is always link 0 and the coupler is marked with an overbar,
- the link rotations that are to be swapped,
- the loops and the path to the coupler point used for the original mechanism,
- for one cognate of the Stephenson-2A where a linear combination of loops is required, it is denoted as (0-1-2-$\hat{4}$-5)+(2-3-$\hat{4}$-5) to indicate that the combination eliminates θ_4 as detailed in Eq. (12),
- for one cognate of the Watt-1B where a linear combination of the path to the coupler point and a loop is required, it is denoted as p=(0-$\hat{3}$-4-5)+(2-$\hat{3}$-4-5) to indicate that the combination eliminates θ_3 which is obtained by simply rescaling the loop and adding it to the path to the coupler point,
- the loops and the path to the coupler point used for the cognate mechanism, given before and after the permutation of rotations, e.g., if rotations 2 and 3 are to be swapped, we might write 0′-1′-2′-3′→0-1-3-2.

Comparing the specifications in this appendix for the four-bar, Stephenson-2A, and Watt-1A to the corresponding derivations in the main body may help clarify the abbreviated notation. One can easily verify that the functions for the original and the cognate mechanism contain the same rotations, so that coefficient matching can be done, and that the number of coefficients to be matched is less than or equal to the number of unknowns (6 link parameters plus one stretch-rotation for the four-bar, 10 link parameters plus 2 stretch-rotations for the six-bars). The "less than" case occurs only for the Watt-1A (as shown in Section 5.3). In all other cases, further analysis of these linear matching conditions shows that they are independent, and hence each permutation corresponds with a unique cognate.

A.1 Four-bar

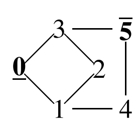
A cognate triple exists (original plus two cognates).



Swap	Original	Cognate
1′2′→21	0-1-2-3	0′-1′-2′-3′→0-2-1-3
	p=0-1-2	p=0′-1′-2′→0-2-1
2′3′→32	0-1-2-3	0′-1′-2′-3′→0-1-3-2
	p=0-3-2	p=0′-3′-2′→0-2-3

A.2 Stephenson-1

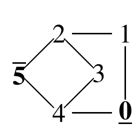
A cognate pair exists.



Swap	Original	Cognate
	0-1-2-3	0′-1′-2′-3′→0-2-1-3
1′2′→21	1-4-5-3-2	1′-4′-5′-3′-2′→2-4-5-3-1
	p=0-3-5	p=0′-3′-5′→0-3-5

A.3 Stephenson-2A

A cognate quadruple exists.

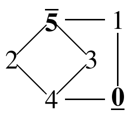


Swap	Original	Cognate
	0-1-2-3-4	0′-1′-2′-3′-4′→0-1-3-2-4
2′3′→32	2-3-4-5	2′-3′-4′-5′→3-2-4-5
	p=0-4-5	p=0′-4′-5′→0-4-5
	0-1-2-5-4	0′-1′-2′-5′-4′→0-1-2-4-5
4′5′→54	2-3-4-5	2′-3′-4′-5′→2-3-5-4
	p=0-4-5	p=0′-4′-5′→0-5-4

Swap	Original	Cognate
$2'3'\to32$ and $4'5'\to54$	2-3-4-5 (0-1-2-3-$\hat{4}$)+(2-3-$\hat{4}$-5) p=0-4-5	$2'$-$3'$-$4'$-$5'$ $\to$3-2-5-4 $0'$-$1'$-$2'$-$3'$-$4'$$\to$0-1-3-2-5 p=$0'$-$4'$-$5'$ $\to$0-5-4

A.4 Stephenson-2B

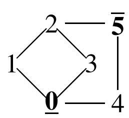
A cognate triple exists.



Swap	Original	Cognate
$2'4'\to42$	0-1-5-2-4 2-4-3-5 p=0-1-5	$0'$-$1'$-$5'$-$2'$-$4'$$\to$0-1-5-4-2 $2'$-$4'$-$3'$-$5'$ $\to$4-2-3-5 p=$0'$-$1'$-$5'$ $\to$0-1-5
$3'4'\to43$	0-1-5-3-4 2-4-3-5 p=0-1-5	$0'$-$1'$-$5'$-$3'$-$4'$$\to$0-1-5-4-3 $2'$-$4'$-$3'$-$5'$ $\to$2-3-4-5 p=$0'$-$1'$-$5'$ $\to$0-1-5

A.5 Stephenson-III

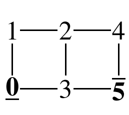
There exists a group of six cognates. As found in [7] and discussed in [14], these can be generated by applying the 3-way Roberts cognates to the four-bar 0-1-2-3 and a 2-way skew pantograph transformation to the dyad 0-4-5. Our method also applies, as summarized below.



Swap	Original	Cognate
$1'2'\to21$	0-1-2-3 0-1-2-5-4 p=0-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-2-1-3 $0'$-$1'$-$2'$-$5'$-$4'$$\to$0-2-1-5-4 p=$0'$-$4'$-$5'$ $\to$0-4-5
$2'3'\to32$	0-1-2-3 0-3-2-5-4 p=0-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-1-3-2 $0'$-$3'$-$2'$-$5'$-$4'$$\to$0-2-3-5-4 p=$0'$-$4'$-$5'$ $\to$0-4-5
$4'5'\to54$	0-1-2-3 0-1-2-5-4 p=0-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-1-2-3 $0'$-$1'$-$2'$-$5'$-$4'$$\to$0-1-2-4-5 p=$0'$-$4'$-$5'$ $\to$0-5-4
$1'2'\to21$ and $4'5'\to54$	0-1-2-3 0-1-2-5-4 p=0-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-2-1-3 $0'$-$1'$-$2'$-$5'$-$4'$$\to$0-2-1-4-5 p=$0'$-$4'$-$5'$ $\to$0-5-4
$2'3'\to32$ and $4'5'\to54$	0-1-2-3 0-3-2-5-4 p=0-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-1-3-2 $0'$-$3'$-$2'$-$5'$-$4'$$\to$0-2-3-4-5 p=$0'$-$4'$-$5'$ $\to$0-5-4

A.6 Watt-1A

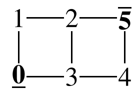
A two-dimensional set of cognates exists.



Swap	Original	Cognate
none	0-1-2-3 2-3-5-4 p=0-3-5	$0'$-$1'$-$2'$-$3'$$\to$0-1-2-3 $2'$-$3'$-$5'$-$4'$$\to$2-3-5-4 p=$0'$-$3'$-$5'$$\to$0-3-5

A.7 Watt-1B

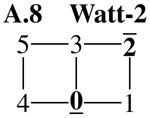
A cognate quadruple exists.



Swap	Original	Cognate
$2'3'\to32$	0-1-2-3 2-3-4-5 p=0-3-2-5	$0'$-$1'$-$2'$-$3'$ $\to$0-1-3-2 $2'$-$3'$-$4'$-$5'$ $\to$3-2-4-5 p=$0'$-$3'$-$2'$-$5'$$\to$0-2-3-5
$4'5'\to54$	0-1-2-3 2-3-4-5 p=0-3-4-5	$0'$-$1'$-$2'$-$3'$ $\to$0-1-2-3 $2'$-$3'$-$4'$-$5'$ $\to$2-3-5-4 p=$0'$-$3'$-$4'$-$5'$$\to$0-3-5-4

Swap	Original	Cognate
$2'3'\to32$ and $4'5'\to54$	0-1-2-3 2-3-4-5 p=(0-$\hat{3}$-4-5)+(2-$\hat{3}$-4-5)	$0'$-$1'$-$2'$-$3'$ $\to$0-1-3-2 $2'$-$3'$-$4'$-$5'$ $\to$3-2-5-4 p=$0'$-$3'$-$4'$-$5'$$\to$0-2-5-4

A.8 Watt-2



The Watt-2 linkage can only draw circles and four-bar curves. The formulas for the four-bar can be applied to loop 0-1-2-3.