Homotopies for connected components of algebraic sets with application to computing critical sets

Daniel J. Bates [*]   Daniel A. Brake [†]   Jonathan D. Hauenstein [‡]

Andrew J. Sommese [§]   Charles W. Wampler [¶]

October 16, 2014

Abstract

Given a polynomial system f, this article provides a general construction for homotopies that yield at least one point of each connected component of the set of solutions of $f = 0$. This approach is then used to compute a superset of the isolated points in the image of an algebraic set which arises in many applications, such as computing critical sets used in the decomposition of real algebraic sets. Examples are presented which demonstrate the efficiency of this approach.

Keywords. Numerical algebraic geometry, polynomial system, algebraic sets, homotopy, projections

AMS Subject Classification. 65H10,68W30,14P05

Introduction

For a polynomial system f with complex coefficients, the fundamental problem of algebraic geometry is to understand the set of solutions of the system $f = 0$, denoted $\mathcal{V}(f)$. *Numerical algebraic geometry* (see, e.g., [5, 23] for a general overview) is based on using homotopy continuation methods for computing $\mathcal{V}(f)$. Geometrically, one can decompose $\mathcal{V}(f)$ into its irreducible components, which corresponds numerically to computing a numerical irreducible decomposition with each irreducible component represented by a witness set. The first step of computing a numerical irreducible decomposition is to compute witness point supersets with the algorithms [11, 20, 22] relying upon

1

a sequence of homotopies. By considering connected components of $\mathcal{V}(f)$ rather than irreducible components, one can use a single homotopy derived from [17, Thm. 7] to compute a finite set of points in $\mathcal{V}(f)$ containing at least one point on each connected component of $\mathcal{V}(f)$. This is complementary to methods for computing a finite set of points in the set of real points in $\mathcal{V}(f)$, denoted $\mathcal{V}_{\mathbb{R}}(f)$, containing at least one point on each connected component of $\mathcal{V}_{\mathbb{R}}(f)$ [1, 9, 19, 27].

Standard homotopy methods applied to f produce a finite set of points in $\mathcal{V}(f)$, including all of the isolated points in $\mathcal{V}(f)$. This new approach permits a similar approach for *numerical elimination theory* [5, Chap. 16]. In particular, suppose that $f(x, y)$ is a polynomial system that is defined on a product of two projective spaces. Let $X = \pi(\mathcal{V}(f))$ under the projection $\pi(x, y) = x$. If $S \subset \mathcal{V}(f)$ is a finite set of points containing a point on each connected component of $\mathcal{V}(f)$, then $\pi(S)$ is a finite set of points in X containing the isolated points of X. The new approach enables one to compute such a set S using a single homotopy; one does not need to separately consider each possible dimension of the fiber over the isolated points of X.

In the classical setting, one can construct the set of isolated solutions from a superset by using, for example, either the global homotopy membership test [21] or the numerical local dimension test [3]. In the elimination setting, a homotopy membership test was developed in [10] building on the membership test of [21] while a local dimension test in this setting remains an open problem.

This approach based on connected components also has many other applications, particularly related to so-called critical point conditions. For example, the methods mentioned above in relation to real solutions, namely [1, 9, 19, 27], compute critical points of $\mathcal{V}(f)$ with respect to the distance function (see also [8]). In [6, 7], critical points of $\mathcal{V}(f)$ with respect to a linear projection are used to numerically decompose real algebraic sets. Other applications include computing witness point sets for irreducible components of rank-deficiency sets [2], isosingular sets [12], and deflation ideals [15].

The rest of the article is organized as follows. The homotopies derived from [17, Thm. 7] are developed in § 1 for computing at least one point on each connected component of $\mathcal{V}(f)$. The extension to elimination theory is presented in § 2, with § 3 focusing on computing critical sets of projections of real algebraic sets. An example illustrating this approach and its efficiency is presented in § 4.

1 Construction of homotopies

Our method of constructing a homotopy for finding at least one point on each connected component is based on [17, Thm. 7]. In this section, we consider the algebraic case of this theorem, and for the convenience of the reader, sketch a proof. We refer to [23] for details regarding algebraic and analytic sets with [17, Appendix] providing a quick introduction to basic results regarding such sets.

Let $\mathcal{E}$ be a complex algebraic vector bundle on an n-dimensional irreducible and reduced complex projective set X. Denote the bundle projection from $\mathcal{E}$ to X by $\pi_{\mathcal{E}}$. A section s of $\mathcal{E}$ is a complex algebraic map $s : X \to \mathcal{E}$ such that $\pi_{\mathcal{E}} \circ s$ is the identity, i.e., $(\pi_{\mathcal{E}} \circ s)(x) = \pi_{\mathcal{E}}(s(x)) = x$ for all $x \in X$.

There is a nonempty Zariski open set $\mathcal{U} \subset X$ over which $\mathcal{E}$ has a trivialization. Using such a trivialization, an algebraic section of $\mathcal{E}$ becomes a system of $\text{rank}(\mathcal{E})$ algebraic functions. In fact, *all* polynomial systems arise in this way and results about special homotopies which track different numbers of paths such as [14, 18, 24] are based on this interpretation (see also [23, Appendix A]).

To help the reader, we specialize this to a concrete situation.

Example 1. *Let $X \subset \prod_{j=1}^{r} \mathbb{P}^{n_j}$ be an irreducible and reduced n-dimensional algebraic subset of a product of projective spaces. For example, X may be an irreducible component of a system of multihomogeneous polynomials in*

$$z_{1,0}, \ldots, z_{1,n_1}, \ldots, z_{r,0}, \ldots, z_{r,n_r},$$

where $z_{j,0}, \ldots, z_{j,n_j}$ are homogeneous coordinates on the jth projective space. Each homogeneous coordinate $z_{j,k}$ has a natural interpretation as a section of the hyperplane section bundle, which we denote by $\mathcal{L}_{\mathbb{P}^{n_j}}(1)$. The dth power of the hyperplane section bundle is denoted by $\mathcal{L}_{\mathbb{P}^{n_j}}(d)$. A multihomogeneous polynomial defined on $\prod_{j=1}^{r} \mathbb{P}^{n_j}$ with multidegree $(d_1, \ldots, d_r)$ is naturally interpreted as a section of the line bundle $\mathcal{L}_{\prod_{j=1}^{r} \mathbb{P}^{n_j}}(d_1, \ldots, d_r) := \otimes_{j=1}^{r} \pi_j^* \mathcal{L}_{\mathbb{P}^{n_j}}(d_j)$, where $\pi_k : \prod_{j=1}^{r} \mathbb{P}^{n_j} \to \mathbb{P}^{n_k}$ is the product projection onto the kth factor. A system of n multihomogeneous polynomials

$$f := \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{1}$$

where f_i has multidegree $(d_{i,1}, \ldots, d_{i,n_i})$ is interpreted as a section of

$$\mathcal{E} := \bigoplus_{i=1}^{n} \mathcal{L}_{\prod_{j=1}^{r} \mathbb{P}^{n_j}}(d_{i,1}, \ldots, d_{i,r}).$$

The solution set of $f = 0$ is simply the set of zeroes of the section f.

We denote the nth Chern class of $\mathcal{E}$, which lies in the $2n^{\text{th}}$ integer cohomology group $H^{2n}(X, \mathbb{Z})$, by $c_n(\mathcal{E})$. Define $d = c_n(\mathcal{E})[X] \in \mathbb{Z}$, i.e., d denotes the evaluation of $c_n(\mathcal{E})$ on X.

Example 2. *Continuing from Example 1, let $c = \sum_{j=1}^{r} n_j - n$ denote the codimension of X. Using multi-index notation for $\alpha = (\alpha_1, \ldots, \alpha_r)$ with each $\alpha_i \geq 0$ and $|\alpha| = \sum_{i=1}^{r} \alpha_i$, we have that X is represented in homology by the sum*

$$\sum_{|\alpha|=c} e_\alpha \mathcal{H}^\alpha$$

where $\mathcal{H}_i := \pi_i^{-1}(H_i)$ with hyperplane $H_i \subset \mathbb{P}^{n_i}$ and $\mathcal{H}^\alpha = \mathcal{H}_1^{\alpha_1} \cdots \mathcal{H}_r^{\alpha_r}$. Moreover, $d := c_n(\mathcal{E})[X]$ is simply the Bézout number of the system of multihomogeneous polynomials restricted to X, i.e., the coefficient of $\prod_{j=1}^{r} z_j^{n_j}$ in the expression

$$\left(\sum_{|\alpha|=c} e_\alpha z^\alpha \right) \cdot \prod_{i=1}^{n} \left(\sum_{j=1}^{r} d_{i,j} z_j \right).$$

This is exactly the number of zeroes of a general section of $\mathcal{E}$ restricted to X.

A vector space V of global sections of $\mathcal{E}$ is said to *span* $\mathcal{E}$ if, given any point $e \in \mathcal{E}$, there is a section $\sigma \in V$ of $\mathcal{E}$ with $\sigma(\pi(e)) = e$. We assume that the rank of $\mathcal{E}$ is $n = \dim X$. If V spans $\mathcal{E}$, then Bertini's Theorem asserts that there is a Zariski dense open set $U \subset V$ with the property that, for all $\sigma \in U$, σ has d nonsingular isolated zeroes contained in the smooth points of X, i.e., the
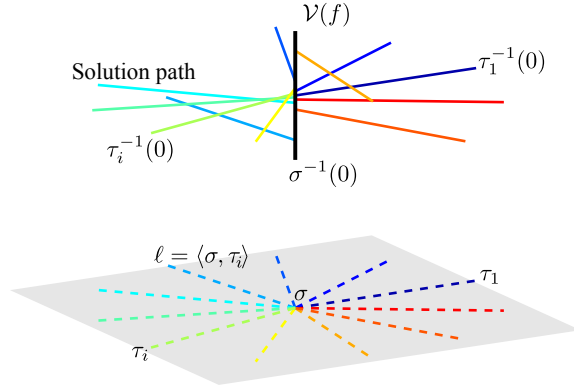
Figure 1: Illustration of the terminology of the paper. The upper space is in terms of the variables of the problem, with solid lines representing solutions paths, starting at the finite nonsingular zeros of τ, and ending at some zero of σ. We show here many τ systems, which all are deformed into σ. At the bottom, the patch represents the vector space V, and the lines ℓ interpolate from τ to σ.

graph of σ meets the graph of the identically zero section of $\mathcal{E}$ transversely in d points in the set of smooth points of X.

Let $|V| := (V \setminus \{0\})/\mathbb{C}^*$ which is the space of lines through the origin of V. Given a complex analytic vector bundle $\mathcal{E}$ spanned by a vector space of complex analytic sections V, the total space $\mathcal{Z} \subset X \times |V|$ of solution sets of $s \in V$ is

$$\mathcal{Z} := \left\{ (x, s) \in X \times |V| \mid s(x) = 0 \right\}.$$

Denote the map of $\mathcal{Z}$ to X induced by the product projection $X \times |V| \to X$ by p and the map of $\mathcal{Z}$ to $|V|$ induced by the projection $X \times |V| \to |V|$ by q.

Since V spans $\mathcal{E}$, the evaluation map

$$X \times V \to \mathcal{E}$$

is surjective, and hence the kernel is a vector bundle of rank $\dim V - \mathrm{rank}(\mathcal{E})$. Let $\mathcal{K}$ denote the dual of this kernel and $\mathbb{P}(\mathcal{K})$ denote $(\mathcal{K}^* \setminus X)/\mathbb{C}^*$, the space of lines through the vector space fibers of the bundle projection of $\mathcal{K}^* \to X$. The convention of denoting $(\mathcal{K}^* \setminus X)/\mathbb{C}^*$ by $\mathbb{P}(\mathcal{K})$ (and not $\mathbb{P}(\mathcal{K}^*)$) is convenient in calculations and followed by a majority of algebraic geometers.

The space $\mathbb{P}(\mathcal{K})$ is easily identified with $\mathcal{Z}$ and the map p is identified with the map $\mathbb{P}(\mathcal{K}) \to X$ induced by the bundle projection. From this identification, we know that $\mathcal{Z}$ is irreducible.

Let $\mathcal{E}$ denote a rank n algebraic vector bundle on a reduced and irreducible projective algebraic set spanned by a vector space V of algebraic sections of $\mathcal{E}$. Suppose that $\sigma \in V$ and $\tau \in V$ have distinct images in $|V|$ and let $\ell := \langle \sigma, \tau \rangle \subset |V|$ denote the unique projective line, i.e., linear $\mathbb{P}^1$, through the images of σ and τ in $|V|$. Letting λ and μ be homogeneous coordinates on ℓ, i.e, spanning sections of $\mathcal{L}_\ell(1)$, we have the section $H(x, \lambda, \mu) := \lambda\sigma + \mu\tau$ of $q^*_{q^{-1}(\ell)}\mathcal{L}_\ell(1) \otimes p^*\mathcal{E}$. Choosing a trivialization of $\mathcal{E}$ over a Zariski open dense set U and a trivialization of $\mathcal{L}_\ell(1)$ over a Zariski open dense set of ℓ, e.g., the set where $\mu \neq 0$, H is naturally interpreted as a homotopy. See Figure 1 for an illustration.

4

With this general setup, we are now able to state and prove the theoretical underpinnings, derived from [17, Thm. 7], for computing a finite set of points containing at least one point on each connected component of $\sigma^{-1}(0)$.

Theorem 3. *Let $\mathcal{E}$ denote a rank n algebraic vector bundle over an irreducible and reduced n-dimensional projective algebraic set X. Let V be a vector space of sections of $\mathcal{E}$ that spans $\mathcal{E}$. Assume that $d := c_n(\mathcal{E})[X] > 0$ and $\tau \in V$ which has d nonsingular zeroes all contained in the smooth points of X. Let $\sigma \in V$ be a nonzero section of $\mathcal{E}$, which is not a multiple of τ. Let ℓ and H be as above. Then, there is a Zariski open set $\mathcal{Q} \subset \ell$ such that*

1. *the map $q_{\mathcal{Z}_\mathcal{Q}}$ of $\mathcal{Z}_\mathcal{Q} := \left\{ H^{-1}(0) \cap (X \times \mathcal{Q}) \right\}$ to ℓ is finite-to-one and*

2. *$\overline{\mathcal{Z}_\mathcal{Q}} \cap \sigma^{-1}(0)$ contains at least one point of every connected subset of $\sigma^{-1}(0)$.*

Proof. The map $q : \mathcal{Z} \to |V|$ may be Stein factorized [23, Thm. A.4.8] as $q = s \circ r$, where $r : \mathcal{Z} \to Y$ is an algebraic map with connected fibers onto an algebraic set Y and $s : Y \to |V|$ is an algebraic map with finite fibers. Since q is surjective, it follows that s is surjective and therefore $\dim Y = \dim |V|$. Since $\mathcal{Z}$ is irreducible, Y is irreducible.

It suffices to show that given any $y \in Y$, there is a complex open neighborhood U of y with $s(U)$ an open neighborhood of $s(y)$. A line $\ell \subset |V|$ is defined by $\dim |V| - 1$ linear equations. Thus, $s^{-1}(\ell)$ has all components of dimension at least 1. The result follows from [23, Thm. A.4.17]. $\square$

Remark 4. Note that if X is an irreducible component of multiplicity one of the solution set of a polynomial system $f_1, \ldots, f_c$ of codimension c in the total space, we can choose our homotopy so that the paths over $(0, 1]$ are in the set where $df_1 \wedge \cdots \wedge df_c$ is non-zero.

2 Isolated points of images

With the theoretical foundation presented in § 1, this section focuses on using it to compute a finite set of points containing at least one point on each connected component in the image of an algebraic set and thus, in particular, a finite superset of the isolated points in the image. Without loss of generality, we may consider projections of algebraic sets. This case corresponds algebraically with computing solutions of an elimination ideal.

Theorem 5. *Let f be a polynomial system defined on $\prod_{j=1}^r \mathbb{P}^{n_j}$ and π denote the projection $\prod_{j=1}^r \mathbb{P}^{n_j} \to \prod_{j=1}^k \mathbb{P}^{n_j}$ onto the first $1 \leq k < r$ spaces. If S is a finite set of points in $\mathcal{V}(f)$ that contains a point on each connected component of $\mathcal{V}(f)$, then $\pi(S)$ is a finite set of points in $\pi(\mathcal{V}(f))$ which contains a point on each connected component of $\pi(\mathcal{V}(f))$. In particular, $\pi(S)$ is a finite superset of the isolated points in $\pi(\mathcal{V}(f))$.*

Proof. Suppose that $C_\pi \subset \pi(\mathcal{V}(f))$ is a connected component. Then, there is an irreducible component $X_\pi \subset \pi(\mathcal{V}(f))$ contained in C_π. Hence, there is an irreducible component $X \subset \mathcal{V}(f)$ such that $\pi(X) = X_\pi$. Let C be the connected component of $\mathcal{V}(f)$ containing X. Thus, $\pi(C) \subset \pi(\mathcal{V}(f))$ is a connected set containing X_π. It immediately follows that $X_\pi \subset \pi(C) \subset C_\pi$ since C_π is the largest connected subset of $\pi(\mathcal{V}(f))$ containing X_π. Since $S \cap C \neq \emptyset$, we have $\pi(S) \cap \pi(C) \subset \pi(S) \cap C_\pi \neq \emptyset$. $\square$

Suppose now that f is a polynomial system defined on $\mathbb{C}^N \times \mathbb{P}^M$. Let $\mathcal{V}(f) \subset \mathbb{C}^N \times \mathbb{P}^M$ and $Z(f) \subset \mathbb{P}^N \times \mathbb{P}^M$ be the closure of $\mathcal{V}(f)$ under the natural embedding of $\mathbb{C}^N$ into $\mathbb{P}^N$. The approach of Theorem 3 can be used to compute a point on each connected component of $Z(f)$. In particular, it may not yield a point on each connected component of $\mathcal{V}(f)$ as the point computed may be at "infinity." One special case is the following for isolated points in the projection of $\mathcal{V}(f)$ onto $\mathbb{C}^N$.

Corollary 6. *Let f be a polynomial system defined on $\mathbb{C}^N \times \mathbb{P}^M$ and π denote the projection $\mathbb{C}^N \times \mathbb{P}^M \to \mathbb{C}^N$. By considering the natural inclusion of $\mathbb{C}^N$ into $\mathbb{P}^N$, let $Z(f)$ be the closure of $\mathcal{V}(f)$ in $\mathbb{P}^N \times \mathbb{P}^M$. If S is a finite set of points in $Z(f)$ that contains a point on each connected component of $Z(f)$, then $\pi(S_\mathbb{C})$ is a finite set of points in $\pi(\mathcal{V}(f))$ which contains the isolated points in $\pi(\mathcal{V}(f))$ where $S_\mathbb{C}$ is the set of points of S contained in $\mathbb{C}^N \times \mathbb{P}^M$.*

Proof. Suppose that $x \in \pi(\mathcal{V}(f)) \subset \mathbb{C}^N$ is isolated. Let $y \in \mathbb{P}^M$ such that $(x,y) \in \mathcal{V}(f)$. By abuse of notation, we have $(x,y) \in Z(f)$ so that there is a connected component, say C, of $Z(f)$ which contains (x,y). Since x is isolated in $\pi(\mathcal{V}(f))$, we must have $C \subset \{x\} \times \mathbb{P}^M$. The statement follows from the fact that C is thus naturally contained in $\mathbb{C}^N \times \mathbb{P}^M$. $\square$

Example 7. *To illustrate the general approach, consider the polynomial system*

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \end{bmatrix}$$

defined on $\mathbb{C}^4$. The set $\mathcal{V}(F) \subset \mathbb{C}^4$ is an irreducible surface of degree six which contains a unique real point, namely the origin, which is an isolated singularity. We can locate this point as follows.

At a singular point of $\mathcal{V}(F)$, the rank of dF falls, and in particular, the two rows of dF are linearly dependent. Accordingly, consider the following system defined on $\mathbb{C}^4 \times \mathbb{P}^1$:

$$G(x,v) = \begin{bmatrix} F(x) \\ v_1 \cdot dF_1(x) + v_2 \cdot dF_2(x) \end{bmatrix}.$$

Since G consists of 6 polynomials defined on a 5 dimensional space, we reduce to a square system via randomization[1] which, for example, yields:

$$\sigma(x,v) := \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^4 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \\ v_1(x_1 + x_4) + v_2(3x_1^2 + x_4) \\ v_1(x_2 + x_4) + v_2(3x_2^2 + x_4) \\ v_1(x_3 + x_4) + v_2(3x_3^2 + x_4) \end{bmatrix}.$$

---

[1]In usual practice, "randomization" means replacing a set of polynomials with some number of random linear combinations of the polynomials. When the appropriate number of combinations is used, then in a Zariski-open subset of the Cartesian space of coefficients of the linear combinations, the solution set of interest is preserved. See, for example, [23, §13.5]. Here, for simplicity of illustration, we take very simple linear combinations involving small integers. These happen to suffice, but in general one would use a random number generator and possibly hundreds of digits to better approximate the probability-one chance of success that is implied in a continuum model of the coefficient space.

As in the discussion in § 1, we use this σ to form a homotopy $H((x,v),\lambda,\mu) = \lambda\sigma(x,v) + \mu\tau(x,v)$, with τ corresponding to the linear product [24] system:

$$\tau(x,v) := \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^4 \\ x_1^3 + x_2^3 + x_3^3 + x_4^2 \\ (v_1 + v_2)(x_1 - 4x_4 - 1)(x_1 - 2) \\ (v_1 - v_2)(x_2 + 2x_4 - 1)(x_2 - 3) \\ (v_1 + 2v_2)(x_3 - 3x_4 - 1)(x_3 - 4) \end{bmatrix}.$$

With this setup, $\tau^{-1}(0)$ has exactly $d = 72$ nonsingular isolated solutions which can be computed easily, for example, using regenerative extension [13] starting with a witness set for $\mathcal{V}(F)$.

We used Bertini *[4] to track the 72 paths along a real arc contained in the line $\ell = \langle \sigma, \tau \rangle$. In this case, 30 paths diverge to infinity and 42 paths end at finite points. Of the latter, 20 endpoints are nonsingular isolated solutions which are extraneous in that they arose from the randomization and not actually in $\mathcal{V}(G)$. The other 22 paths converged to points in $\{0\} \times \mathbb{P}^1$, 18 of which ended with $v = [0,1] \in \mathbb{P}^1$, while the other 4 break into 2 groups of 2 with v of the form $[1,\alpha]$ and $[1, \text{conj}(\alpha)]$ where $\alpha \approx -0.351 + 0.504 \cdot \sqrt{-1}$. In particular, even though $\{0\} \times \mathbb{P}^1$ is a positive-dimensional solution component of $\mathcal{V}(\sigma)$ and also of $\mathcal{V}(G)$, we always obtain at least one point on this component and thus observe that the origin is singular.*

3   Computing critical points of projections

One special case of Corollary 6 is when X is an irreducible curve of the solution set $\mathcal{V}(f)$ of a polynomial system $f := \{f_1, \ldots, f_{N-1}\}$ defined on $\mathbb{C}^N$ which has multiplicity one. Let $\pi : X \to \mathbb{C}$ be a linear projection. A critical point of π with respect to X is a point $x \in X$ such that either

- x is a smooth point and $d\pi$ is zero on the tangent space of X at x; or

- x is a singular point of X.

In [7], which includes an implementation of the curve decomposition algorithm of [16], we need to compute the finite set of critical points of π that are isolated points of the subset of X not meeting other components of $\mathcal{V}(f)$. In fact, the approach only needs to a finite superset of such points. The extra points that are not critical points simply make the cellular decomposition of [7] finer, which can be merged away in a post-processing step. Thus, we need to find a superset of the isolated points of the solution set of $x \in X$ not meeting other components of $\mathcal{V}(f)$ and such that

$$\text{rank} \begin{bmatrix} d\pi \\ df_1 \\ \vdots \\ df_{N-1} \end{bmatrix} \leq N - 1. \tag{2}$$

Table 1: Decomposition of 12-bar spherical linkage.

dimension	degree	# components
3	8	2
2	4	2
	8	14
	12	12
	16	1
	20	4
	24	1
1	4	6
	6	2

It suffices to find at least one point in each connected component of the solution set of

$$
\begin{bmatrix}
\begin{array}{c} f_1 \\ \vdots \\ f_{N-1} \end{array} \\
\begin{bmatrix} d\pi \\ df_1 \\ \vdots \\ df_{N-1} \end{bmatrix} \cdot \xi
\end{bmatrix} = 0
$$

where $\xi \in \mathbb{P}^{N-1}$. The advantage here is that we obtain a finite superset of the critical points using one homotopy regardless of the possibly different dimension of the corresponding null spaces, i.e., there is no need to cascade down the possible null space dimensions.

The setup above naturally extends to computing witness point supersets for the critical set of dimension $k - 1$ of an irreducible component of dimension k, e.g., critical curves of a surface.

4   Example

Consider the 12-bar spherical linkage from [25, 26]. (The device can be viewed as 20 rigid rods meeting in spherical joints at 9 points, or since a loop of three such rods forms a rigid triangle, as 12 rigid links meeting in rotational hinges with the axes of rotation all intersecting at a central point. The arrangement is most clearly seen in Figure 2(c).) The irreducible components of the variety for the mechanism, which is contained in $\mathbb{C}^{18}$, are summarized in Table 1 which was first computed in [9]. Take $X \subset \mathbb{C}^{18} \times \mathbb{P}^{17}$ to be the union of the eight one-dimensional irreducible components, which has degree 36, crossed with $\mathbb{P}^{17}$. We will use our new approach with Bertini v1.4 [4] to compute a superset of the real critical points of the real projection defined in (3).

The 18 variables for the polynomial system F defining the 12-bar spherical linkages under consideration are $P_{11}, P_{12}, P_{13}, \ldots, P_{61}, P_{62}, P_{63}$ with $P_i = (P_{i1}, P_{i2}, P_{i3})$ corresponding to the location of the ith point. The ground link is fixed by taking $P_0 = (0, 0, 0)$, $P_7 = (-1, 1, -1)$, and

$P_8 = (-1, -1, -1)$. With this setup, $F = \{G_{ij}, H_k\}$ which consists of 17 polynomials:

$$G_{ij} = |P_i - P_j|^2 - 4$$
$$(i, j) \in \{(1,2), (3,4), (5,6), (1,5), (2,6), (3,7), (4,8), (1,3), (2,4), (5,7), (6,8)\};$$
$$H_k = |P_k|^2 - 3, \quad k \in \{1, 2, 3, 4, 5, 6\}.$$

To compute a superset of the real critical points with respect to the real projection

$$\pi(P) = \frac{3}{5}P_{11} + \frac{13}{17}P_{12} - \frac{5}{16}P_{13} + \frac{26}{27}P_{21} - \frac{1}{10}P_{22} + \frac{1}{6}P_{23} + \frac{3}{5}P_{31} + \frac{7}{17}P_{32} + \frac{3}{10}P_{33}+$$
$$\frac{1}{4}P_{41} - \frac{4}{5}P_{42} + \frac{1}{3}P_{43} + \frac{18}{25}P_{51} + \frac{14}{29}P_{52} - \frac{12}{13}P_{53} - \frac{17}{30}P_{61} - \frac{5}{17}P_{62} + \frac{13}{20}P_{63}, \qquad (3)$$

we consider the following system defined on $X \subset \mathbb{C}^{18} \times \mathbb{P}^{17}$:

$$f(P, \xi) = \begin{bmatrix} F(P) \\ \begin{bmatrix} d\pi \\ dF(P) \end{bmatrix} \cdot \xi \end{bmatrix}.$$

Since the irreducible curves of $\mathcal{V}(F)$ have multiplicity one, we know that the irreducible components of $\mathcal{V}(f) \cap X$ must be of the form $\{x\} \times L$ for some point $x \in \mathcal{V}(f)$ and linear space $L \subset \mathbb{P}^{17}$. In this particular case, we only aim to compute all such points x. However, before using the new approach, we consider some other options for this computation. These other approaches do yield additional information, namely witness point supersets for the irreducible components. One option is to consider each possible dimension of $\mathbb{P}^{17}$ independently. Since the zero-dimensional case is equivalent in terms of the setup and number of paths to the new approach discussed below, we will just quickly summarize without performing the full computation. For each $0 \leq i \leq 16$, starting with a witness set for X, the corresponding start system, after possible randomization, would require tracking $36 \cdot (17 - i)$, totaling 5508, paths related to moving linear slices and the same number of paths to compute witness point supersets.

Rather than treat each dimension independently, another option is to cascade down through the dimensions, e.g., using the regenerative extension [13]. The implementation in `Bertini`, starting with a witness set for X, requires tracking 6276 paths for solving as well as tracking 3216 paths related to moving linear slices. Using 64, 2.3 GHz processors, this computation took 618 seconds.

Instead of using a method designed for computing witness point supersets, our new approach uses one homotopy to compute a point on each connected component. This is all that is needed for the current application via Corollary 6. Since $d\pi$ is constant and $dF(P)$ is a matrix with linear entries, we take our start system to be

$$g(P, \xi) = \begin{bmatrix} F(P) \\ \xi_0 \\ \ell_1(P) \cdot \xi_1 \\ \vdots \\ \ell_{17}(P) \cdot \xi_{17} \end{bmatrix}$$

which we restrict to X where each ℓ_i is a random linear polynomial. In particular, $\mathcal{V}(g) \cap X$ consists of $d = 36 \cdot 17 = 612$ points, each of which is nonsingular with respect to g. Starting with a witness
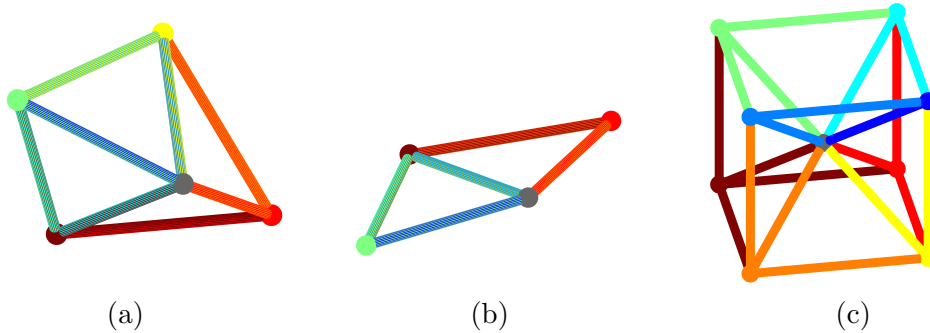
9

Figure 2: Solutions to the 12 bar spherical linkage obtained from the critical point computation: (a) an equilateral spherical four-bar configuration, corresponding to a nonsingular critical point on a degree four irreducible component; (b) a degenerate configuration, coming from the intersection of such a component with a higher-dimensional irreducible component; (c) a rigid configuration arising from the intersection of the irreducible curves of degree six.

set for the X, we can compute $\mathcal{V}(g) \cap X$ by tracking 612 paths related to moving linear slices. Then, we can compute a point on each connected component of $\mathcal{V}(f) \cap X$ by tracking 612 paths. This computation in total, using the same parallel setup as above, took 20 seconds.

Of the 612 paths, 120 converge to finite endpoints, while the rest diverge to infinity. Of the 120 finite endpoints of the form (P, ξ), 78 are real (i.e., have $P \in \mathbb{R}^{18}$), but there are only 22 distinct real points. This is because some points appear with multiplicity, while others have a null space with dimension greater than one so that the same P can appear with several different null directions, ξ. In detail, the breakdown of the 22 real points is as follows:

- 14 real points are the endpoint of one path each. These points had rank $dF = 17$, so they are smooth points on X. These are each on one of the degree 4 irreducible components of X, and each is an equilateral spherical four-bar of the type illustrated in Figure 2(a).

- 2 real points are the endpoint of 2 paths each. In both cases, the two endpoints having the same P are identical, as must be because although rank $dF = 16$, they have rank $\left[\begin{array}{c} d\pi \\ dF \end{array} \right] = 17$, so the null vector is unique in projective space. These points correspond to a rigid arrangement as shown in Figure 2(c), one the mirror image of the other.

- 6 real points are the endpoint of 10 paths each. These points had rank $dF = 12$ with rank $\left[\begin{array}{c} d\pi \\ dF \end{array} \right] = 13$. These points occur where some one of the irreducible components of degree 4 intersects another irreducible component. At these points, the 12-bar appears as in Figure 2(b).

To clarify the accounting, note that $14 \cdot 1 + 2 \cdot 2 + 6 \cdot 10 = 78$.

10

5   Conclusion

We have constructed a new approach for constructing one homotopy that yields a finite superset of solutions to a polynomial system containing at least one point on each connected component of the solution set. This idea naturally leads to homotopies for solving elimination problems, such as computing critical points of projections as well as other rank-constraint problems. This method allows one to compute such points directly without having to cascade through all the possible dimensions of the auxiliary variables. This can provide considerable computational savings, as we have demonstrated on an example arising in kinematics, where the endpoints of a single homotopy include all the critical points on a curve even though the associated null-spaces at these points have various dimensions.

References

[1] P. Aubry, F. Rouillier, and M. Safey El Din. Real solving for positive dimensional systems. *J. Symbolic Comput.*, 34(6):543–560, 2002.

[2] D.J. Bates, J.D. Hauenstein, C. Peterson, and A.J. Sommese. Numerical decomposition of the rank-deficiency set of a matrix of multivariate polynomials. In *Approximate commutative algebra*, Texts Monogr. Symbol. Comput., Springer, 2009, pages 55–77.

[3] D.J. Bates, J.D. Hauenstein, C. Peterson, and A.J. Sommese. A numerical local dimensions test for points on the solution set of a system of polynomial equations. *SIAM J. Numer. Anal.*, 47(5):3608–3623, 2009.

[4] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini: Software for numerical algebraic geometry. Available at `bertini.nd.edu`.

[5] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. *Numerically solving polynomial systems with Bertini*. SIAM, 2013.

[6] D.A. Brake, D.J. Bates, W. Hao, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini_real: Software for one- and two-dimensional real algebraic sets. *LNCS*, 8592:175–182, 2014.

[7] D.A. Brake, D.J. Bates, W. Hao, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Bertini_real: Numerical decomposition of real algebraic curves and surfaces. In preparation, 2014.

[8] J. Darisma, E. Horobet, G. Ottaviani, B. Sturmfels, and R.R. Thomas. The Euclidean distance degree of an algebraic variety. `arXiv:1309.0049`, 2013.

[9] J.D. Hauenstein. Numerically computing real points on algebraic sets. *Acta Appl. Math.*, 125(1):105–119, 2013.

[10] J.D. Hauenstein and A.J. Sommese. Membership tests for images of algebraic sets by linear projections. *Appl. Math. Comput.*, 219(12):6809–6818, 2013.

[11] J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Regenerative cascade homotopies for solving polynomial systems. *Appl. Math. Comput.*, 218(4):1240–1246, 2011.

[12] J.D. Hauenstein and C.W. Wampler. Isosingular sets and deflation. *Found. Comp. Math.*, 13(3):371–403, 2013.

[13] J.D. Hauenstein and C.W. Wampler. Numerical algebraic intersection using regeneration. Preprint available at `www.nd.edu/~jhauenst/preprints`.

[14] B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.

[15] A. Leykin. Numerical primary decomposition. In *ISSAC 2008*, ACM, New York, 2008, pages 165–172.

[16] Y. Lu, D.J. Bates, A.J. Sommese, and C.W. Wampler. Finding all real points of a complex curve. *Contemp. Math.*, 448:183–205, 2007.

[17] A.P. Morgan and A.J. Sommese. Coefficient-parameter polynomial continuation. *Appl. Math. Comput.*, 29(2):123–160, 1989. Errata: *Appl. Math. Comput.* 51:207, 1992.

[18] A.P. Morgan and A.J. Sommese. A homotopy for solving general polynomial systems that respects m-homogeneous structures. *Appl. Math. Comput.*, 24:101–113, 1987.

[19] F. Rouillier, M.-F. Roy, and M. Safey El Din. Finding at least one point in each connected component of a real algebraic set defined by a single equation. *J. Complexity*, 16(4):716–750, 2000.

[20] A.J. Sommese and J. Verschelde. Numerical homotopies to compute generic points on positive dimensional algebraic sets. *J. Complexity*, 16(3):572–602, 2000.

[21] A.J. Sommese, J. Verschelde, and C.W. Wampler. Numerical irreducible decomposition using projections from points on the components. *Contemp. Math.*, 286:37–51, 2001.

[22] A.J. Sommese and C.W. Wampler. Numerical algebraic geometry. In *The mathematics of numerical analysis (Park City, UT, 1995)*, volume 32 of *Lectures in Appl. Math.*, AMS, Providence, RI, 1996, pages 749–763.

[23] A.J. Sommese and C.W. Wampler. *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific, Singapore, 2005.

[24] J. Verschelde and R. Cools. Symbolic homotopy construction. *Appl. Algebra Engrg. Comm. Comput.*, 4(3):169–183, 1993.

[25] C.W. Wampler, B. Larson, and A. Edrman. A New Mobility Formula for Spatial Mechanisms. In *Proc. DETC/Mechanisms & Robotics Conf.*, ASME, 2007, paper DETC2007-35574.

[26] C.W. Wampler, J.D. Hauenstein, and A.J. Sommese. Mechanism mobility and a local dimension test. *Mech. Mach. Theory*, 46(9):1193–1206, 2011.

[27] W. Wu and G. Reid. Finding points on real solution components and applications to differential polynomial systems. In *ISSAC '13*, ACM, New York, 2008, pages 339–346.