Lecture No. 30

Upwinding for the FE method, Petrov-Galerkin Methods

Reference: Carey and Oden, Vol. VI

Westerink and Shea: IJNME, 1989

Cantekin and Westerink, IJNME 1990

In general upwinding for the FEM is achieved by modifying the weighting functions.

Traditional Petrov-Galerkin Methods:

Modify basis functions by a function 1 degree higher than the basis functions.

For linear elements: Quadratic upwinding

$$w_1 = \phi_1 - \alpha F_1(\xi)$$

$$w_2 = \phi_2 + \alpha F_1(\xi)$$

where

$$F_1(\xi) = \frac{3}{4}(\xi+1)(1-\xi)$$

$F_1(\xi)$ is a quadratic function which is zero at the 2 nodes in our linear element. We will call this an N + 1 degree upwinding method.

- see Figure L30.1a for plot of the resulting weighting functions.

- Using the resulting upwind *biased* weighting functions leads to:

$$\underline{M}\underline{u}_{,t} + (\underline{A} + \underline{B})\,\underline{u} = P$$

where the elemental matrices are defined as:

$$\underline{M}^{(n)} = \frac{\Delta x}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \alpha\frac{\Delta x}{4}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\underline{A}^{(n)} = \frac{V^{(n)}}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{\alpha V^{(n)}}{2}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

$$\underline{B}^{(n)} = \frac{D^{(n)}}{\Delta x}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

- Leading terms are standard terms whereas 2nd term is associated with the upwind bias.
- Thus the upwind bias portion of the weighting function generates additional terms for the mass and convection matrices.
- We note that the additional convection matrix really looks identical to the diffusion matrix.
- Like upwinding in FD's, this upwinding causes artificial numerical damping. However in a consistent (non-lumped) formulation, the truncation error is still $0\,(h^2)\,\frac{d^3 u}{dx^3}$ and thus upwinding in FE's is not as bad as in FD's where a truncation error of $0\,(h)\,\frac{d^2 u}{dx^2}$ results which is a diffusion term similar to the physical diffusion term. We note that the lumped upwinded FE formulation with linear elements leads to identical equations as obtained with an upwinded FD formulation.

Linear elements with cubic upwinding

For linear elements:

$$w_1 = \phi_1 - \beta F_2(\xi)$$

$$w_2 = \phi_2 + \beta F_2(\xi)$$

where

$$F_2(\xi) = \frac{5}{8}\xi(\xi+1)(\xi-1)$$

$F_2(\xi)$ is a cubic function which is zero at both nodes in our element. This function is two degrees greater than the interpolating basis and thus we refer to this as an N+2 degree upwind method.

- See Fig. L30.1b for illustration of these weighting functions.
- Use of these functions leads to:

$$\underline{M}\underline{u}_{,t} + (\underline{A} + \underline{B})\,\underline{u} = \underline{P}$$

where the elemental matrices are defined as:

$$\underline{M}^{(n)} = \frac{\Delta x}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\beta\Delta x}{24}\begin{bmatrix} -1 & 1 \\ +1 & -1 \end{bmatrix}$$

$$\underline{A}^{(n)} = \frac{V^{(n)}}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{B}^{(n)} = \frac{D^{(n)}}{\Delta x}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

- We note that only the time derivative term is affected and that this effect would disappear if a lumped formulation were used. Also this suggests that the importance/effect of N+2 upwinding decreases with lower $\mathbf{C}_{\#}$.

Combined N+1/N+2 degree upwinding

$$w_1 = \phi_1 - \alpha F_1(\xi) - \beta F_2(\xi)$$

$$w_2 = \phi_2 + \alpha F_1(\xi) + \beta F_2(\xi)$$

This leads to the following elemental matrices:

$$\underline{M}^{(n)} = \frac{\Delta x}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\alpha \Delta x}{4}\begin{bmatrix} -1 & -1 \\ +1 & +1 \end{bmatrix} + \frac{\beta \Delta x}{24}\begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$\underline{A}^{(n)} = \frac{V^{(n)}}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{\alpha V}{2}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

$$\underline{B}^{(n)} = \frac{D^{(n)}}{\Delta x}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

- This leads to a global system of equations with equation at node i:

$$u_{i-1,j+1}\left[\frac{1}{6\Delta t} + \frac{\alpha}{4\Delta t} + \frac{\beta}{24\Delta t} - \frac{V}{4\Delta x} - \frac{\alpha V}{4\Delta x} - \frac{D}{2\Delta x^2}\right]$$

$$+ u_{i-1,j}\left[-\frac{1}{6\Delta t} - \frac{\alpha}{4\Delta t} - \frac{\beta}{24\Delta t} - \frac{V}{4\Delta x} - \frac{\alpha V}{4\Delta x} - \frac{D}{2\Delta x^2}\right]$$

$$+ u_{i,j+1}\left[\frac{2}{3\Delta t} - \frac{\beta}{12\Delta t} + \frac{\alpha V}{2\Delta x} + \frac{D}{\Delta x^2}\right]$$

$$+ u_{i,j}\left[-\frac{2}{3\Delta t} + \frac{\beta}{12\Delta t} + \frac{\alpha V}{2\Delta x} + \frac{D}{\Delta x^2}\right]$$

$$+ u_{i+1,j+1}\left[\frac{1}{6\Delta t} - \frac{\alpha}{4\Delta t} + \frac{\beta}{24\Delta t} + \frac{V}{4\Delta x} - \frac{\alpha V}{4\Delta x} - \frac{D}{2\Delta x^2}\right]$$

$$+ u_{i+1,j}\left[-\frac{1}{6\Delta t} + \frac{\alpha}{4\Delta t} - \frac{\beta}{24\Delta t} + \frac{V}{4\Delta x} - \frac{\alpha V}{4\Delta x} - \frac{D}{2\Delta x^2}\right] = 0$$

- Truncation error analysis leads to:

$$\tau = \Delta x^2 \left[(2\mathbf{C}_\#^2 - \beta) - \frac{1}{\mathbf{P_e}}(12\alpha) \right] \frac{V}{24} \frac{\partial^3 u}{\partial x^3}$$

$$+ \Delta x^3 \left[-\frac{\mathbf{C}_\#}{2}(2\mathbf{C}_\#^2 - \beta) + \alpha(1 - \mathbf{C}_\#^2) + \frac{1}{\mathbf{P_e}}(2 - 6\mathbf{C}_\#^2 + \beta + 6\alpha\mathbf{C}_\#) \right] \frac{V}{24} \frac{\partial^4 u}{\partial x^4}$$

$$+ \Delta x^4 \left[\left(-\frac{2}{15} + \frac{1}{3}\mathbf{C}_\#^2 + \frac{3}{10}\mathbf{C}_\#^4 - \frac{1}{12}\beta - \frac{1}{6}\beta\mathbf{C}_\#^2\right) \right.$$

$$- \alpha\frac{\mathbf{C}_\#}{2}(1 - \mathbf{C}_\#^2) + \frac{1}{\mathbf{P_e}}(-\mathbf{C}_\# + 3\mathbf{C}_\#^2 - \beta\mathbf{C}_\# - 2\alpha)$$

$$\left. - \frac{1}{\mathbf{P_e^2}}(6\mathbf{C}_\#^2 - 6\mathbf{C}_\#\alpha) \right] \frac{V}{24} \frac{\partial^5 u}{\partial x^5} + H.O.T.\text{'s}$$

- Consider the case $\mathbf{P_e} = \infty \Rightarrow \frac{1}{\mathbf{P_e}} = 0$

Now the leading order truncation term becomes:

$$\Delta x^2 (2\mathbf{C}_\#^2 - \beta)$$

thus we let

$$\beta = 2\mathbf{C}_\#^2$$

This then reduces the truncation error to:

$$\tau = \Delta x^3 [\alpha(1 - \mathbf{C}_\#^2)] \frac{V}{24} \frac{\partial^4 u}{\partial x^4}$$

$$+ \Delta x^4 \left[\left(-\frac{2}{15} + \frac{1}{6}\mathbf{C}_\#^2 - \frac{1}{30}\mathbf{C}_\#^4\right) - \alpha\frac{\mathbf{C}_\#}{2}(1 - \mathbf{C}_\#^2) \right] \frac{V}{24} \frac{\partial^5 u}{\partial x^5}$$

$$+ H.O.T.\text{'s}$$

- Thus in general, to eliminate the $0(\Delta x^3)$ truncation term, we must choose:

$$\alpha = 0$$

We note that when $\mathbf{C}_{\#} = 1.0$ both Δx^3 and Δx^4 truncation terms disappear, regardless of the value of α.

Truncation error analysis conclusions:

1. FE N+1 upwinding is similar to FD upwinding but for a consistent formulation is much better.
2. N+2 upwinding with no N+1 upwinding should be much better than standard or N+1 upwinding. This is due to N+2 upwinding's excellent ability to eliminate truncation error terms.

Fourier Analysis

- $\mathbf{P_e} = \infty$

Standard solution

- Fig. L30.2a,b: perfect amplitude and ratio behavior.
- Fig. L30.2c: we do have phase lag which causes wiggles.

N+1 upwinded solution

- Fig. L30.3a,b: dampens the short wavelengths. However overdamping only occurs at relatively short wavelength/Δx values; not across a very broad range like FD upwinding.
- Fig. L30.3c: slightly better phase behavior than standard but not by much.

N+2 upwinded solution

- Fig. L30.4a,b: perfect amplitude and ratio behavior.
- Fig. L30.4c: much better phase behavior than either N+1 upwind or standard solution.

- $\mathbf{P_e} = 2$

  General conclusion (Figs. L30.5-L30.7): neither N+1 nor N+2 upwinding really effect the solution much.

Numerical Examples

All examples at $\mathbf{P_e} = \infty$

Fig. L30.8a	$\mathbf{C_\#} = 0.24$	$\alpha = 0$	$\beta = 0$
Fig. L30.8b	$\mathbf{C_\#} = 0.24$	$\alpha = 0.20$	$\beta = 0$
Fig. L30.8c	$\mathbf{C_\#} = 0.24$	$\alpha = 0$	$\beta = 0.30$
Fig. L30.9a	$\mathbf{C_\#} = 0.80$	$\alpha = 0$	$\beta = 0$
Fig. L30.9b	$\mathbf{C_\#} = 0.80$	$\alpha = 1.0$	$\beta = 0$
Fig. L30.9c	$\mathbf{C_\#} = 0.80$	$\alpha = 0$	$\beta = 1.28$
Fig. L30.10a	$\mathbf{C_\#} = 1.0$	$\alpha = 0$	$\beta = 0$
Fig. L30.10b	$\mathbf{C_\#} = 1.0$	$\alpha = 1.0$	$\beta = 0$
Fig. L30.10c	$\mathbf{C_\#} = 1.0$	$\alpha = 0$	$\beta = 2.0$
Fig. L30.10d	$\mathbf{C_\#} = 1.0$	$\alpha = 1.0$	$\beta = 2.0$

Notes:

1. Optimal β values experimentally determined in the figures match formula $\beta = 2\mathbf{C}_{\#}^2$ well at high $\mathbf{C}_{\#}$ but deviate at low $\mathbf{C}_{\#}$. This is due to the competition of the various order truncation terms at low $\mathbf{C}_{\#}$.
2. At $\mathbf{C}_{\#} = 1.0$ we get a perfect solution and the α selection does <u>not</u> influence our solution.

<u>Lagrange Quadratic Elements</u>

Now we let

$$w_1 = \phi_1 - \alpha_c F_3(\xi) - \beta_c F_4(\xi)$$

$$w_2 = \phi_1 + 4\alpha_m F_3(\xi) + 4\beta_m F_4(\xi)$$

$$w_3 = \phi_1 - \alpha_c F_3(\xi) - \beta_c F_4(\xi)$$

where

$$F_3(\xi) = \frac{5}{8}\xi(\xi+1)(\xi-1) = F_2(\xi)$$

$$F_4(\xi) = \frac{21}{16}(-\xi^4+\xi^2)$$

- See Figures L30.11a/b for figures of upwinded functions.

- We note that due to their distinctly different nature, corner and mid-element nodes are differently biased.

- These upwinding functions lead to the following elemental matrices:

$$\underline{M}^{(n)} = \frac{\Delta x}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \frac{\Delta x}{120}\begin{bmatrix} -10\alpha_c & 0 & 10\alpha_c \\ 40\alpha_m & 0 & -40\alpha_c \\ -10\alpha_c & 0 & 10\alpha_c \end{bmatrix}$$

$$+ \frac{\Delta x}{120}\begin{bmatrix} -9\beta_c & -24\beta_c & -9\beta_c \\ 36\beta_m & 96\beta_m & 36\beta_m \\ -9\beta_c & -24\beta_c & -9\beta_c \end{bmatrix}$$

$$\underline{A}^{(n)} = \frac{V}{6}\begin{bmatrix} -3 & 4 & -1 \\ -4 & 0 & 4 \\ 1 & -4 & 3 \end{bmatrix} + \frac{V}{120}\begin{bmatrix} 20\alpha_c & -40\alpha_c & 20\alpha_c \\ -80\alpha_m & 160\alpha_m & -80\alpha_m \\ 20\alpha_c & -40\alpha_c & 20\alpha_c \end{bmatrix}$$

$$+ \frac{V}{120}\begin{bmatrix} 21\beta_c & 0 & -21\beta_c \\ -84\beta_m & 0 & 84\beta_m \\ 21\beta_c & 0 & -21\beta_c \end{bmatrix}$$

$$\underline{B}^{(n)} = \frac{D}{6\Delta x}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \frac{D}{20\Delta x}\begin{bmatrix} 7\beta_c & -14\beta_c & 7\beta_c \\ -28\beta_m & 56\beta_m & -28\beta_m \\ 7\beta_c & -14\beta_c & 7\beta_c \end{bmatrix}$$

Numerical Examples

- $\mathbf{P_e} = \infty$ for all examples.

- Fig. L30.12 shows that for $\mathbf{C_\#} = 0.24$ for the standard, N+1 upwinded and N+2 upwinded all solutions are very good.

  Thus quadratic elements are spatially very accurate and don't have oscillatory problems at low $\mathbf{C_\#}$ values as linear elements.

- Fig. L30.13 shows $\mathbf{C_\#} = 0.80$ case.
  - standard solution is poor.
  - N+1 upwinded solution is worse.
  - N+2 upwinded solution is excellent!

Fig L30.1a

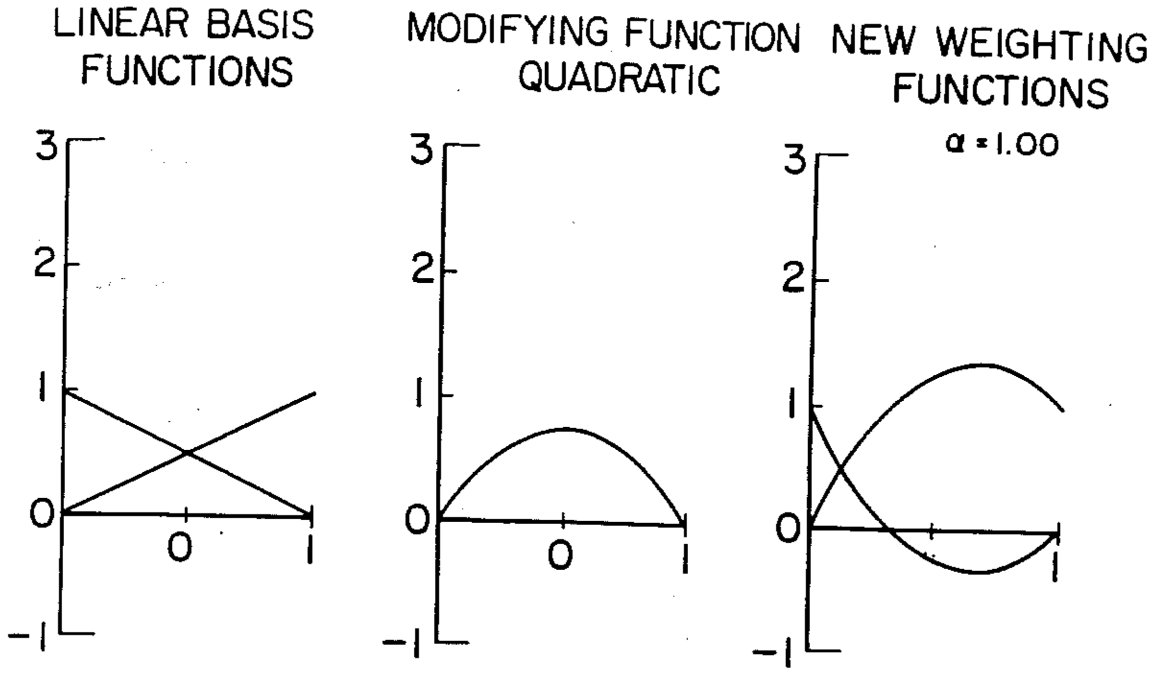


Fig L30.1b

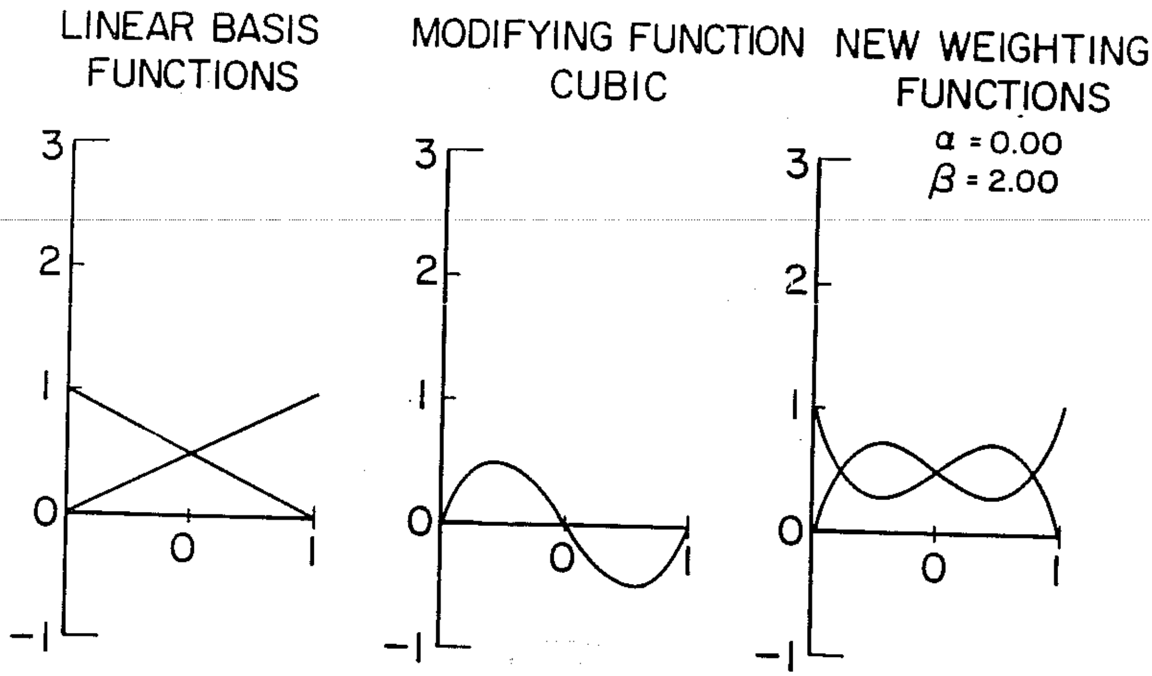


Fig L30.2a

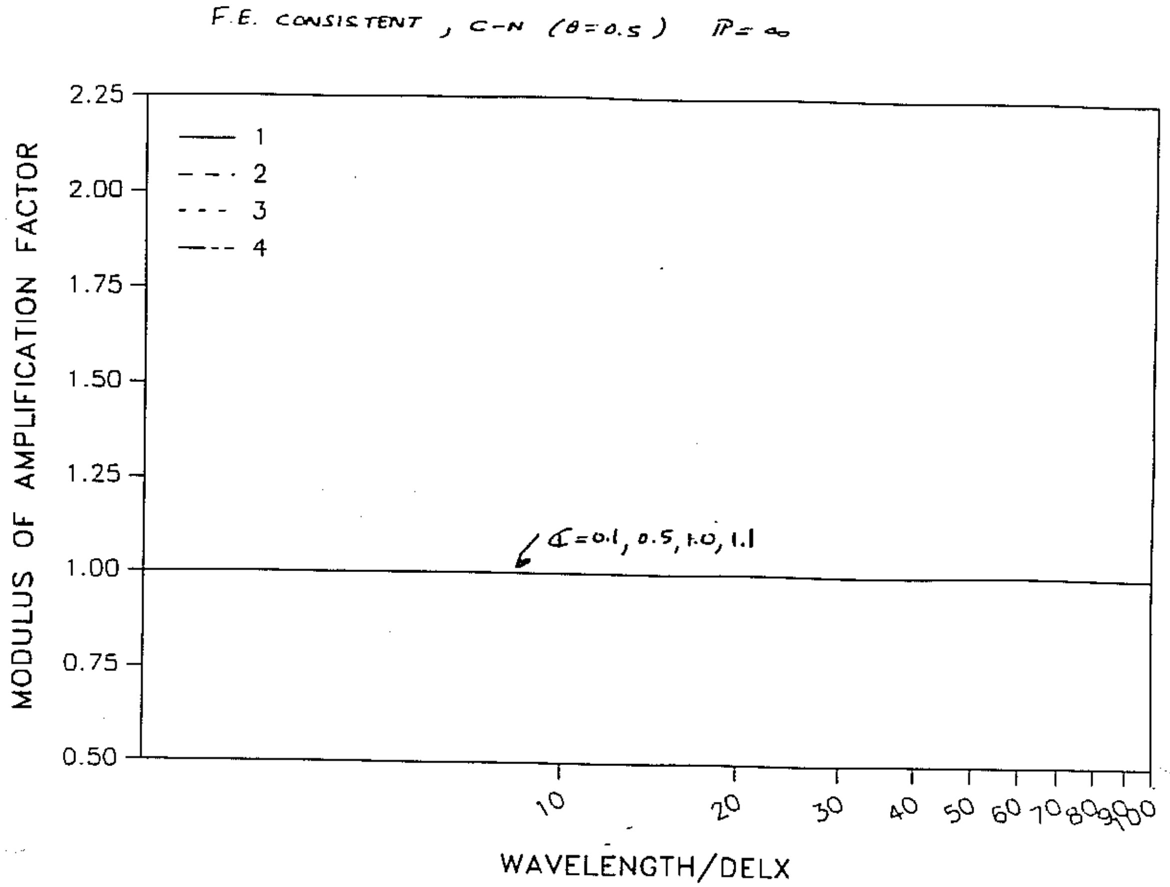
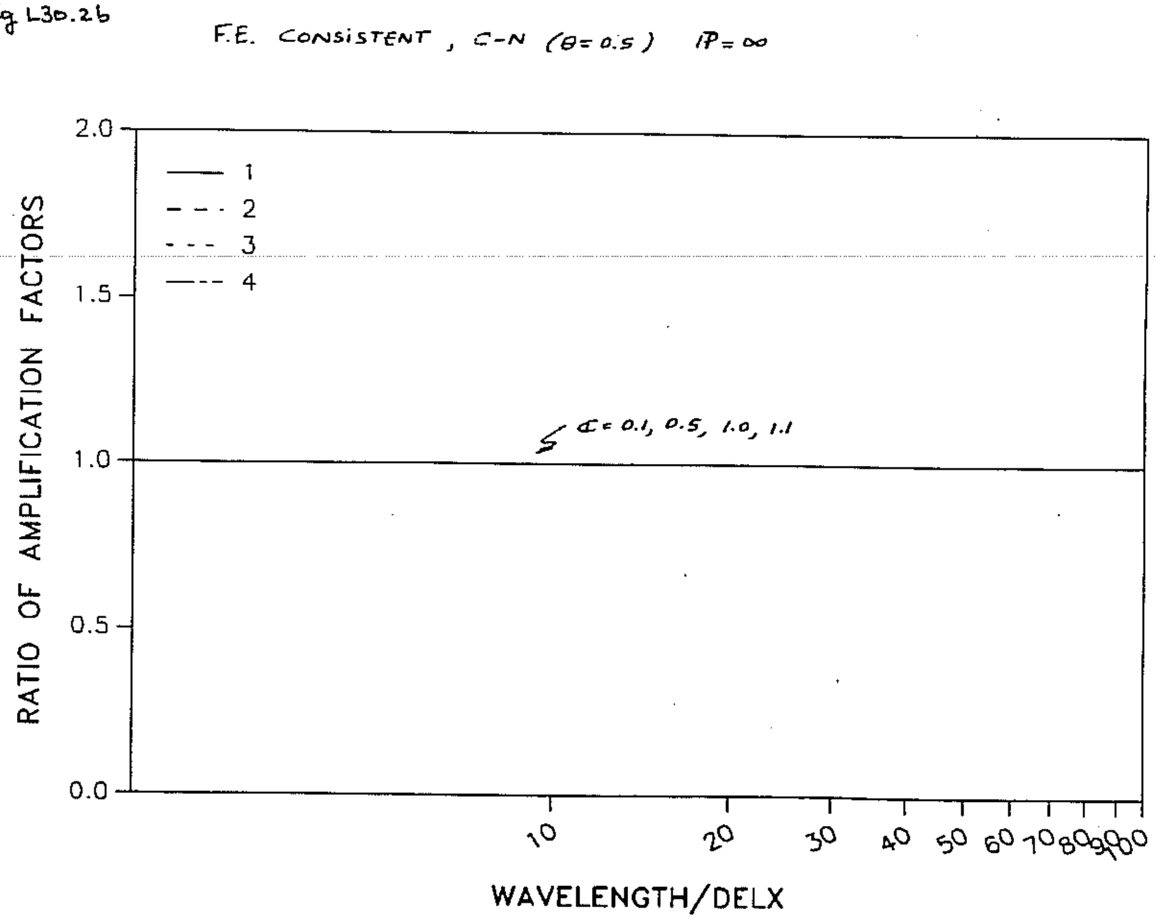


Fig L30.2b



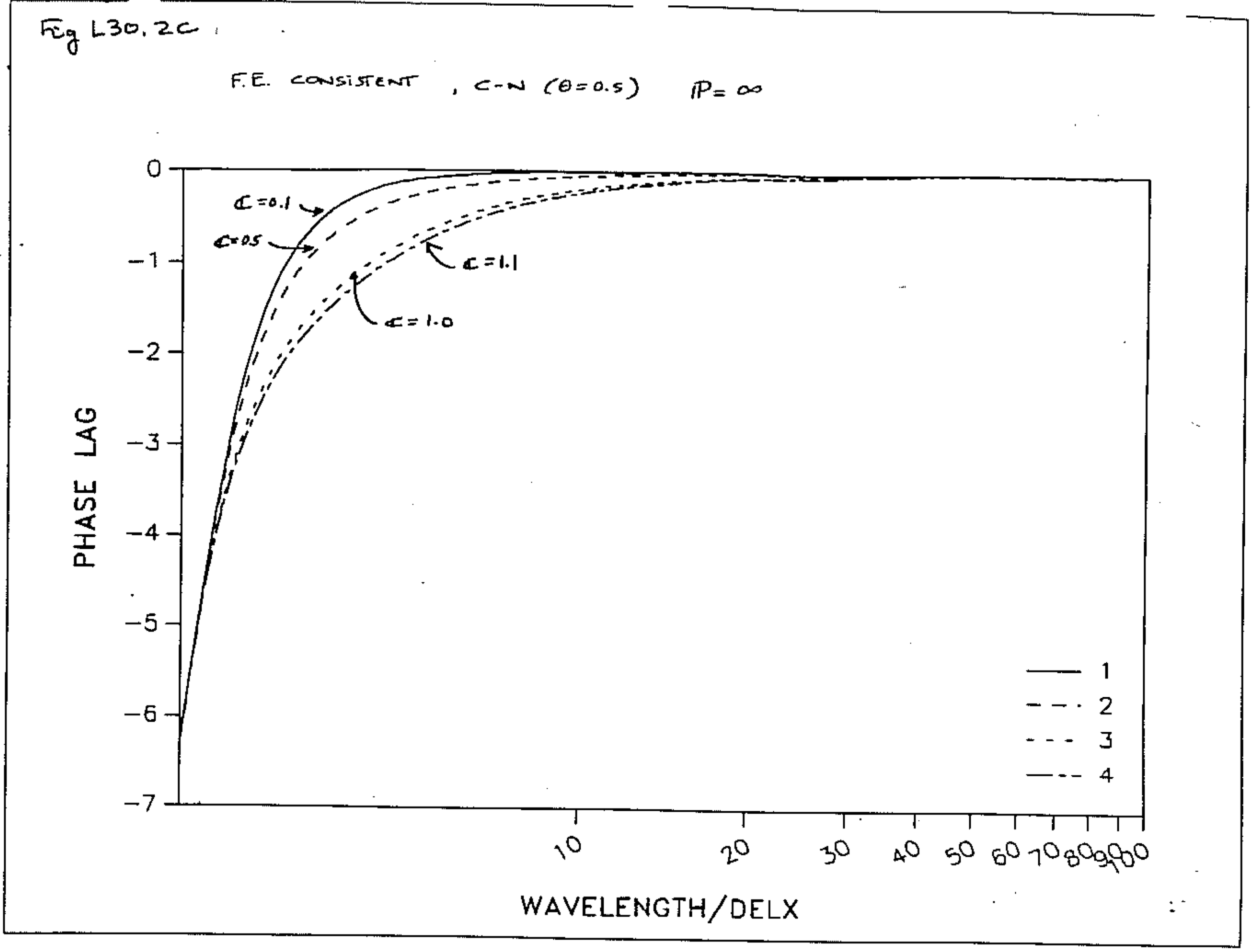


Fig L30.2c
F.E. CONSISTENT, C-N (θ=0.5) IP=∞
PHASE LAG
C=0.1
C=0.5
C=1.1
C=1.0
1
2
3
4
WAVELENGTH/DELX

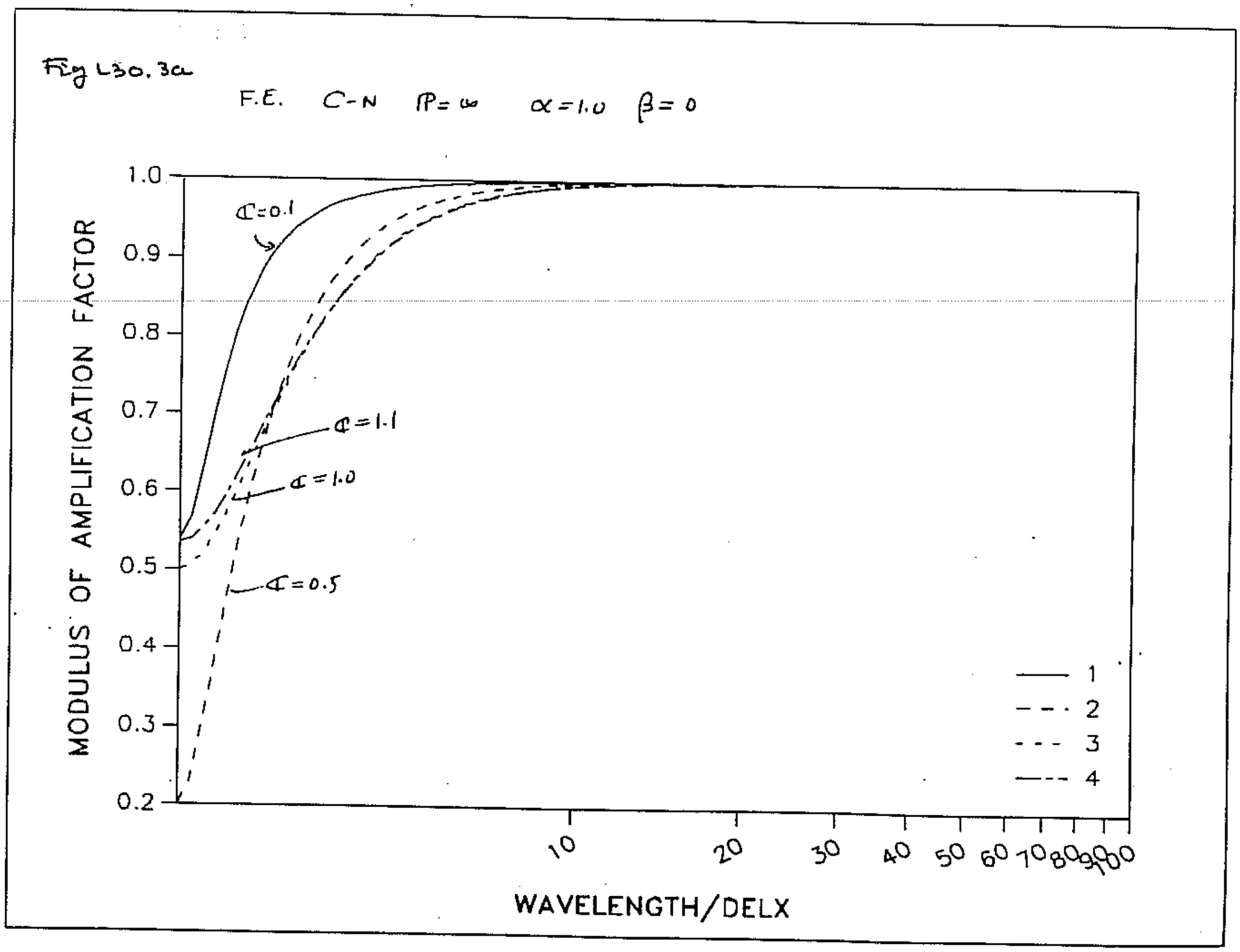


Fig L30.3a
F.E. C-N IP=∞ α=1.0 β=0
MODULUS OF AMPLIFICATION FACTOR
C=0.1
C=1.1
C=1.0
C=0.5
1
2
3
4
WAVELENGTH/DELX

Fig L30.3b F.E - C-N P = ∞ $\alpha = 1.0$ $\beta = 0.0$

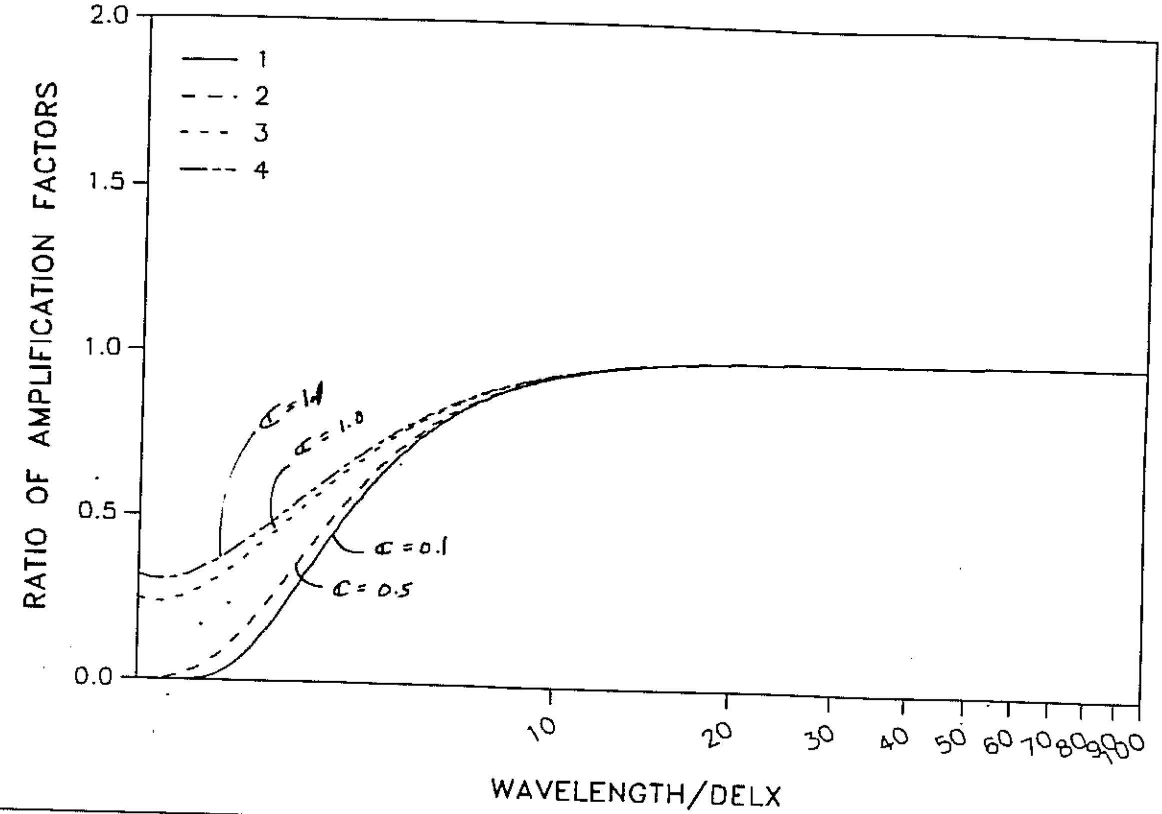
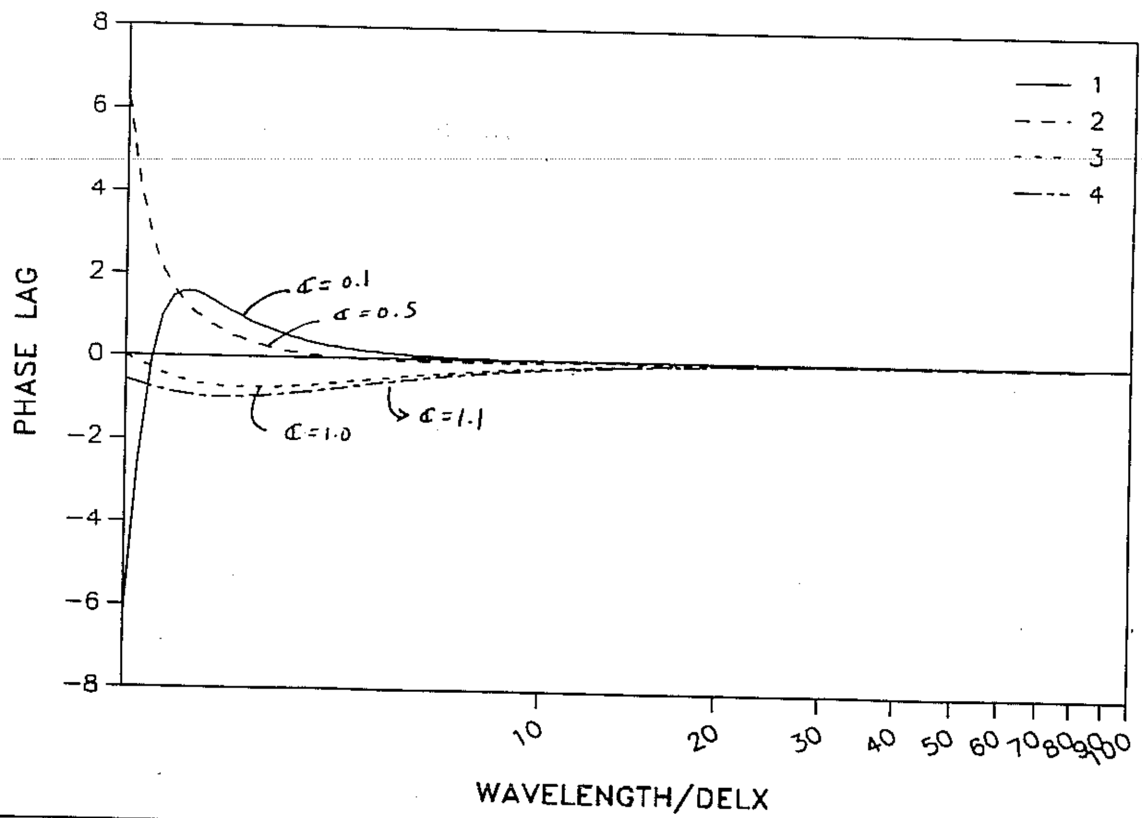


Fig L30.3c F.E. C-N P = ∞ $\alpha = 1.0$ $\beta = 0.0$



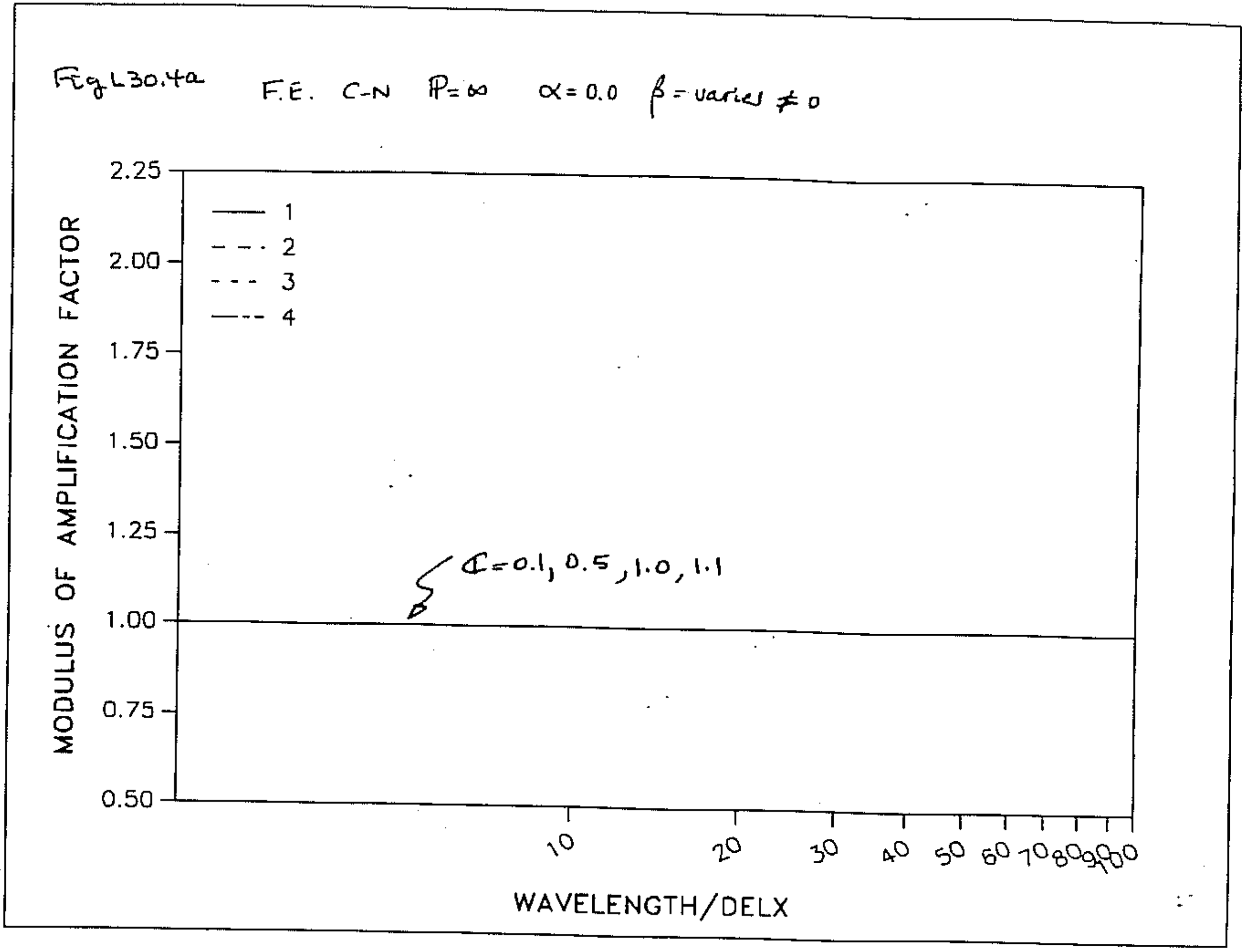


Fig L30.4a F.E. C-N P=∞ α=0.0 β=varies ≠ 0
2.25
2.00
1.75
1.50
1.25
1.00
0.75
0.50
MODULUS OF AMPLIFICATION FACTOR
1
2
3
4
C=0.1, 0.5, 1.0, 1.1
10
20
30
40
50
60
70
80
90
100
WAVELENGTH/DELX

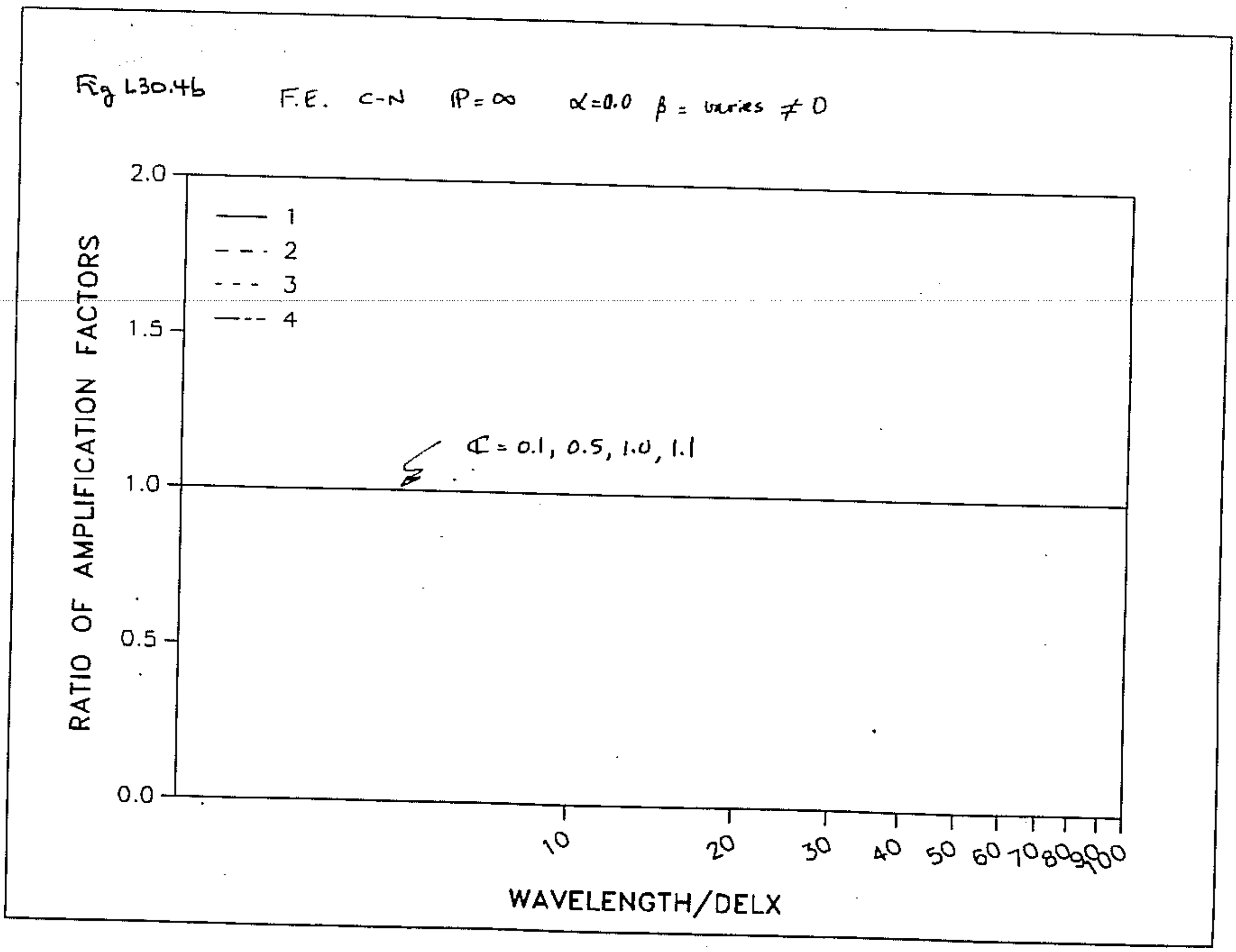


Fig L30.4b F.E. C-N P=∞ α=0.0 β=varies ≠ 0
2.0
1.5
1.0
0.5
0.0
RATIO OF AMPLIFICATION FACTORS
1
2
3
4
C=0.1, 0.5, 1.0, 1.1
10
20
30
40
50
60
70
80
90
100
WAVELENGTH/DELX

Fig L30.4c F.E. C-N P= ∞ α=0.0 β=varies ≠ 0

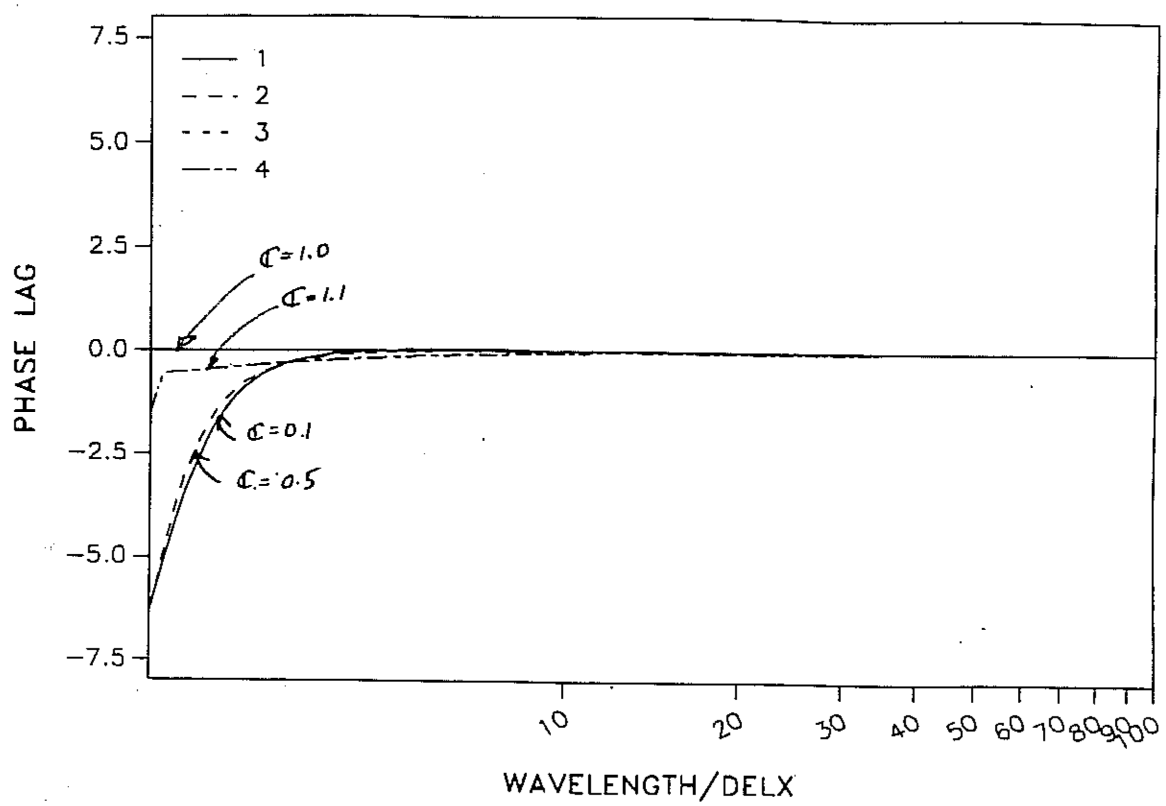


Fig L30.5a F.E. CONSISTENT C-N (θ=0.5) P=2

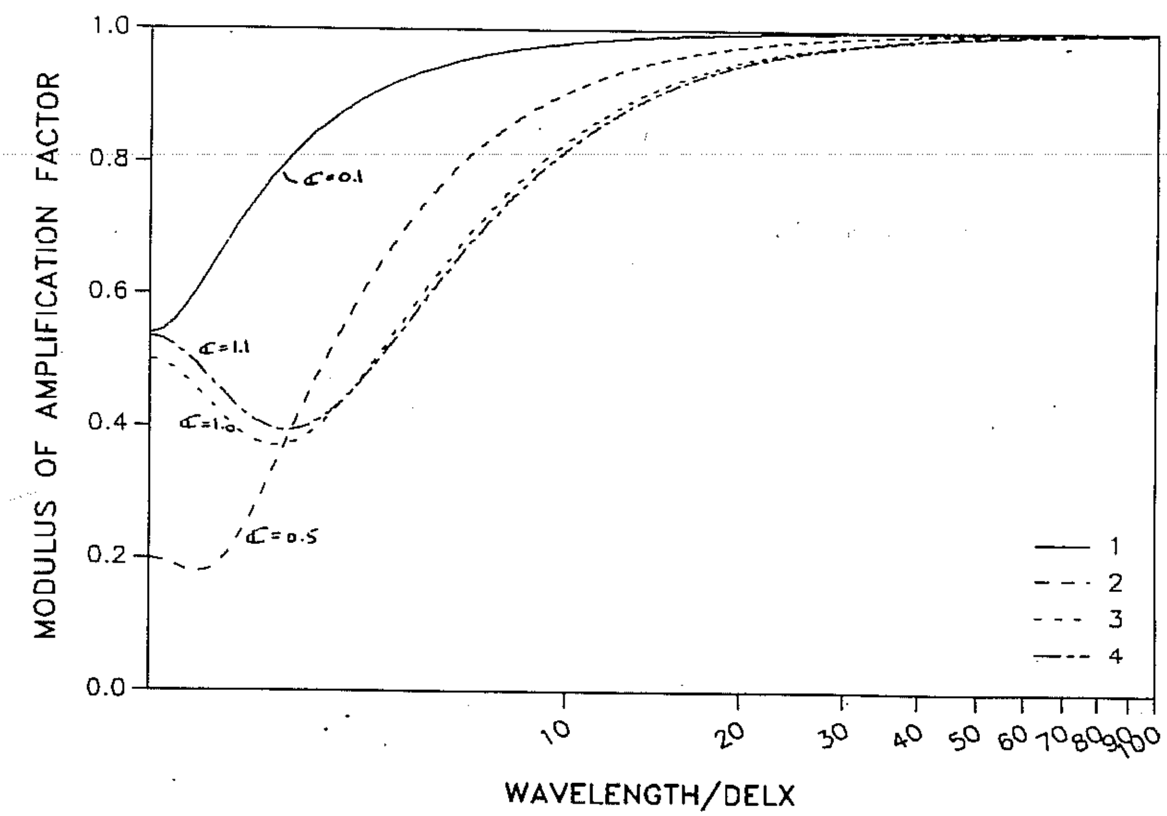


Fig L30.5b

F.E. CONSISTENT , C-N (θ=0.5) IP=2

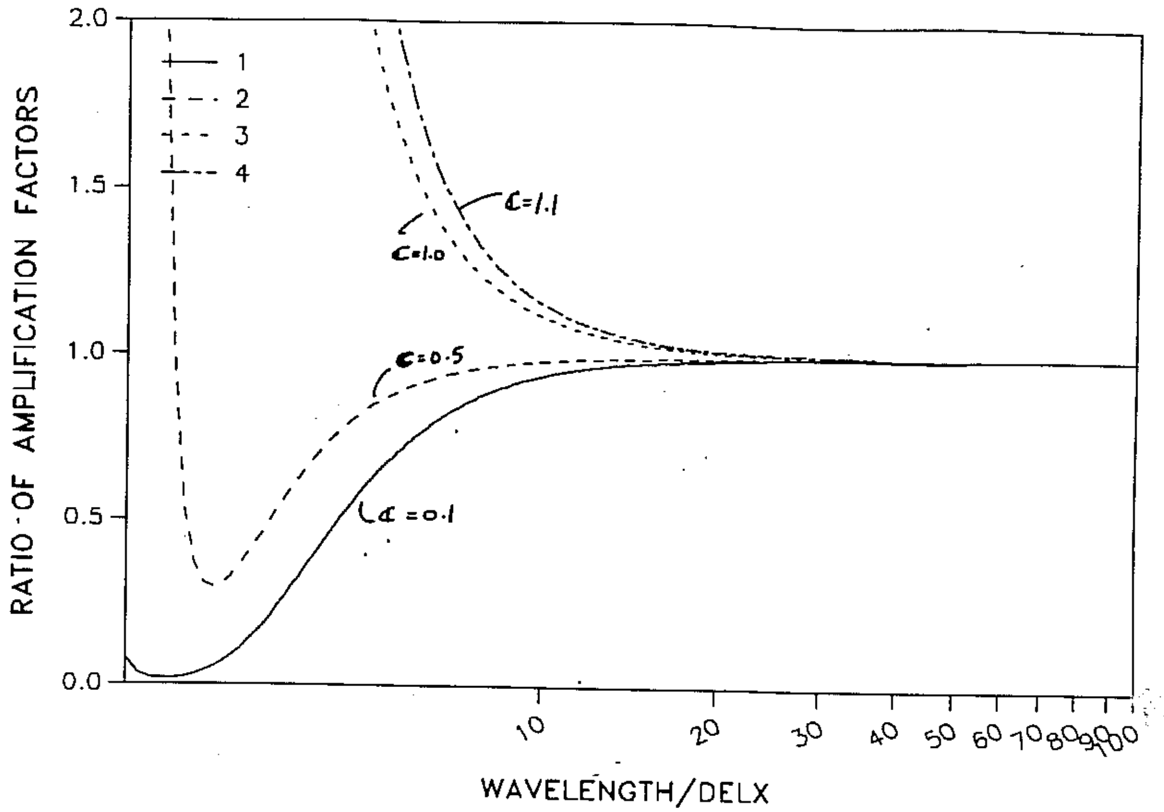


Fig L30.5c

F.E. CONSISTENT , C-N (θ=0.5) IP=2

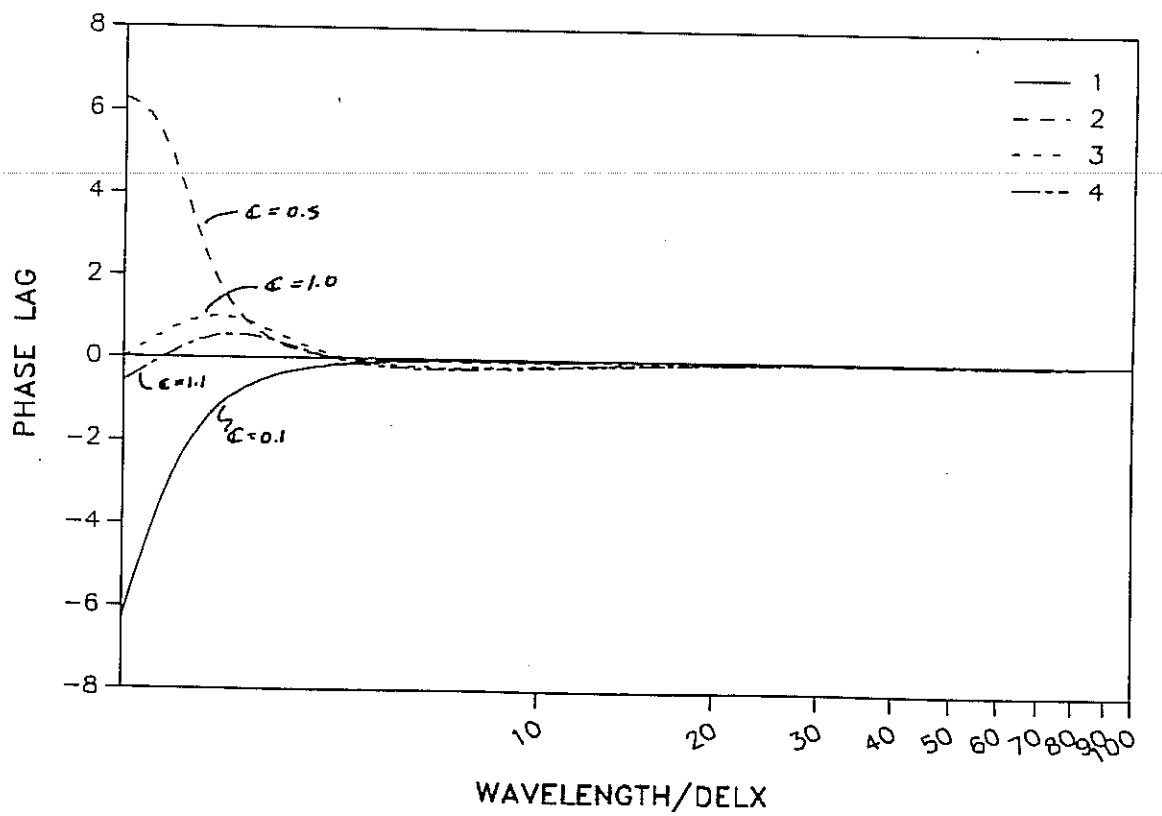


Fig L30.6a

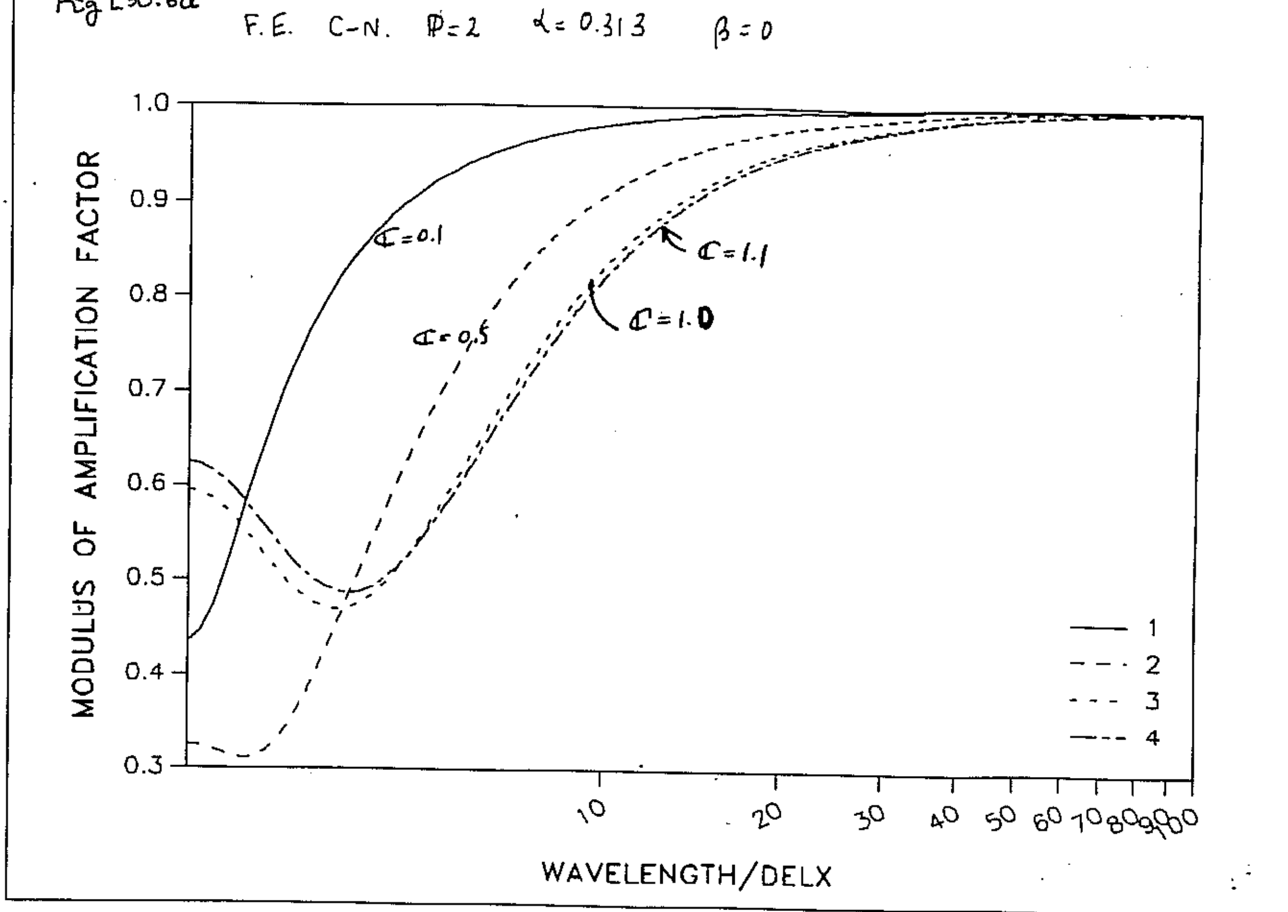


Fig L30.6b

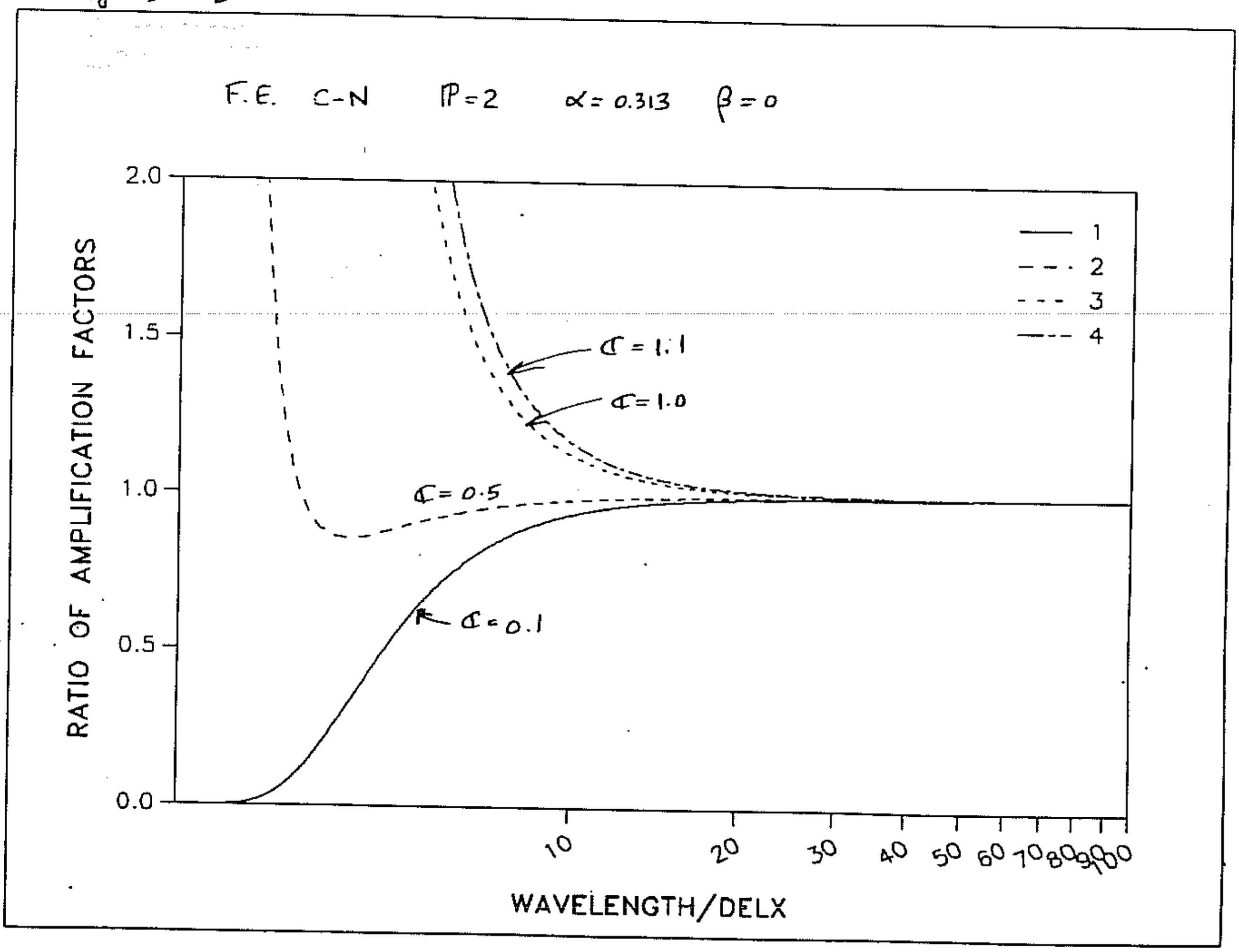


Fig L30.6c

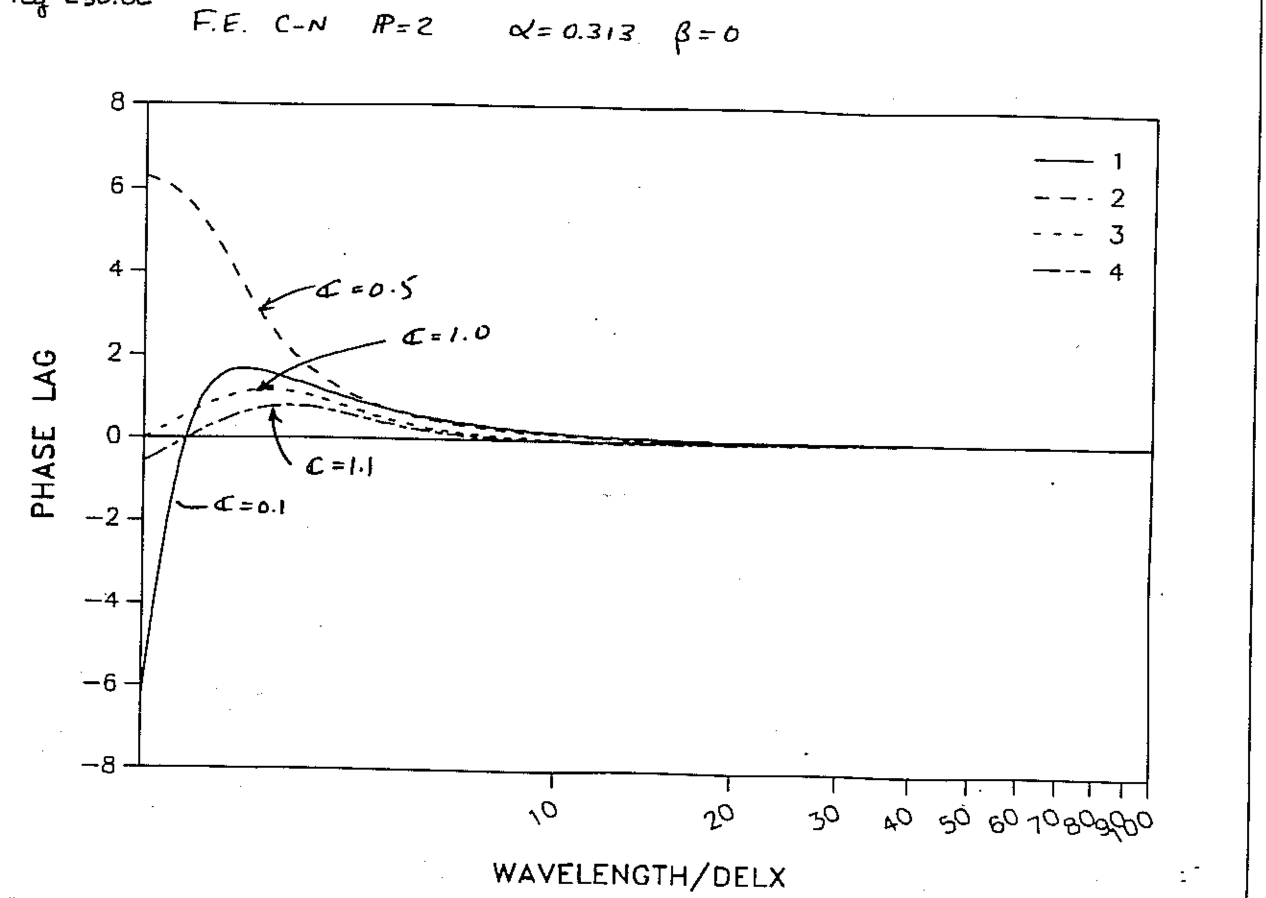


Fig L30.7a

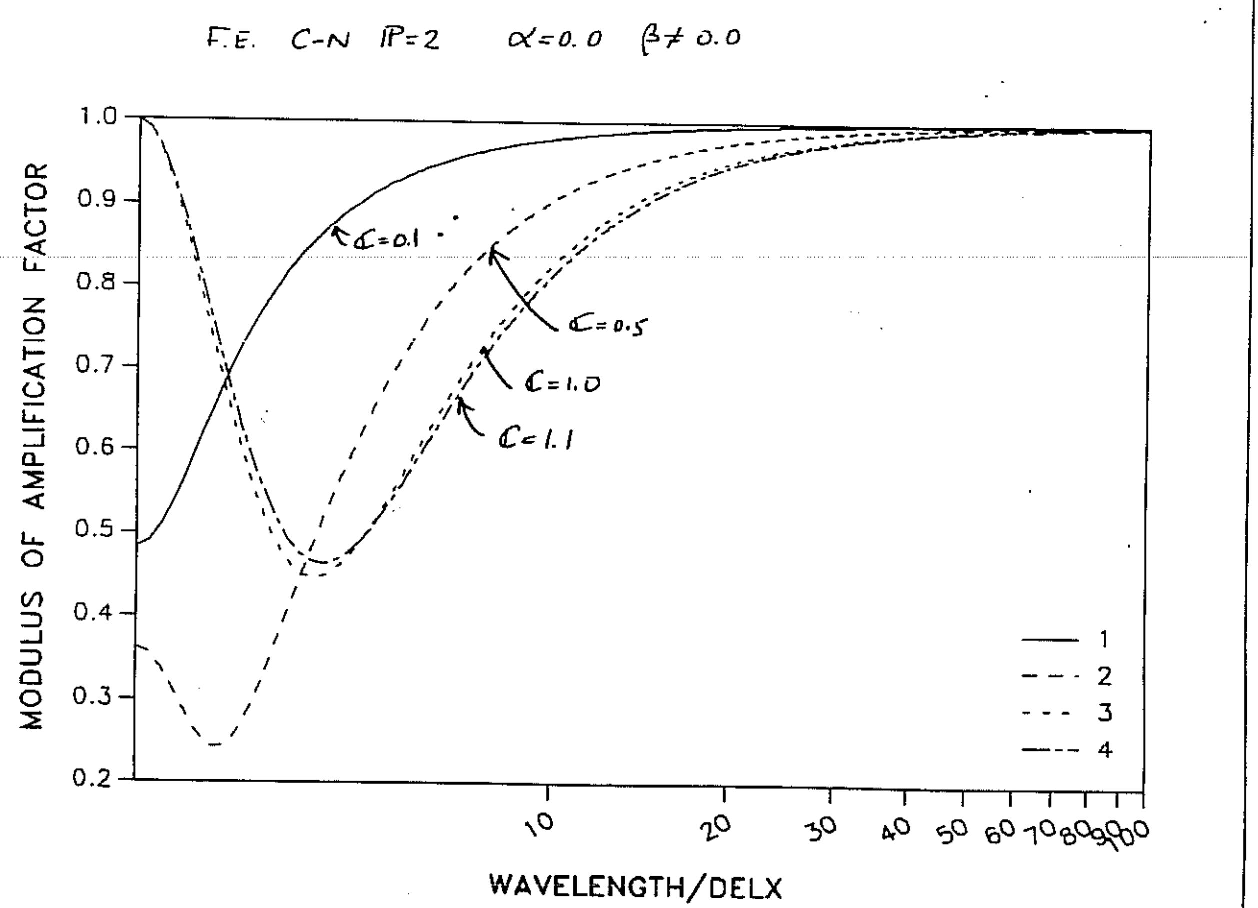


Fig L30.7b

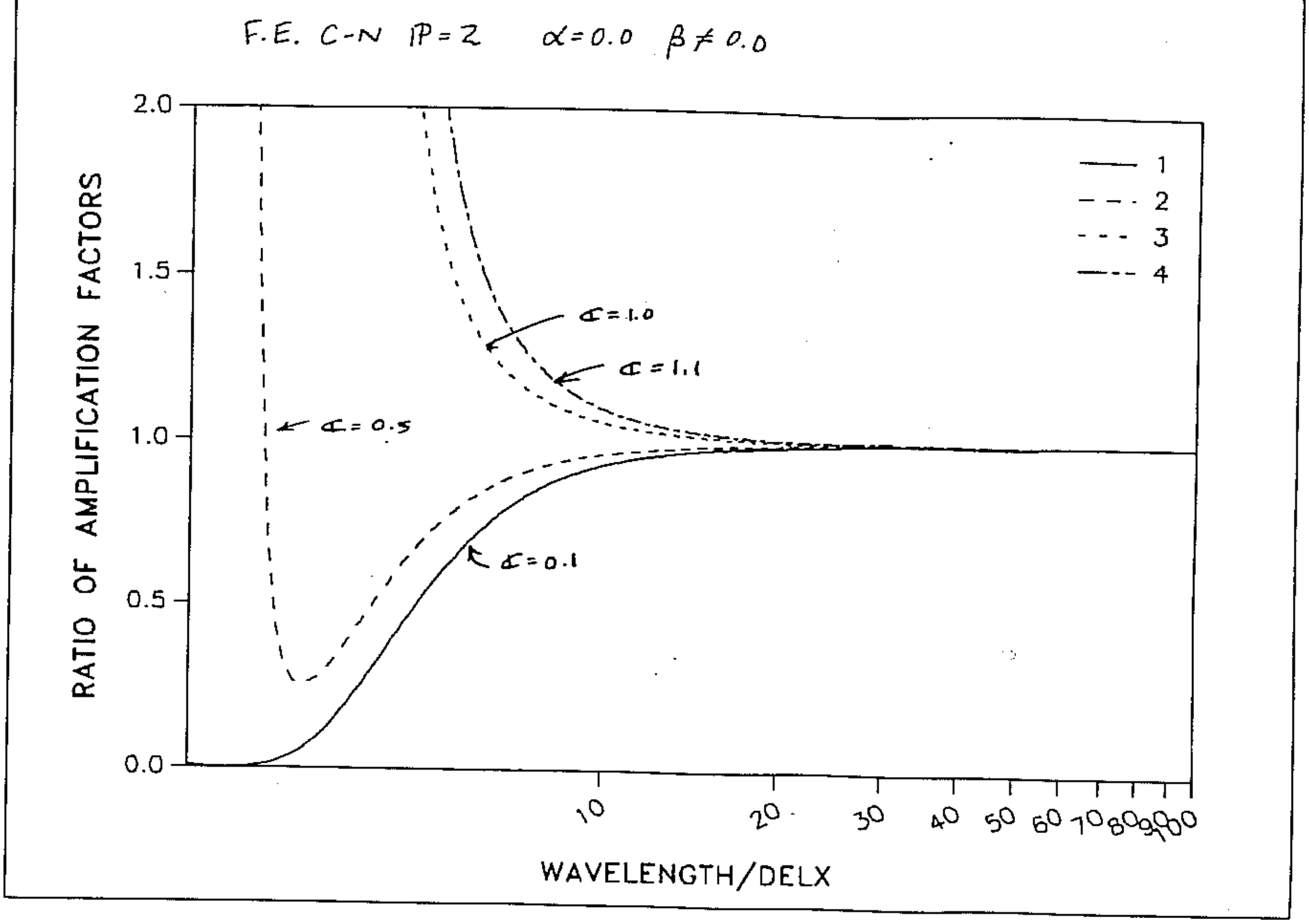


Fig. L30.7c

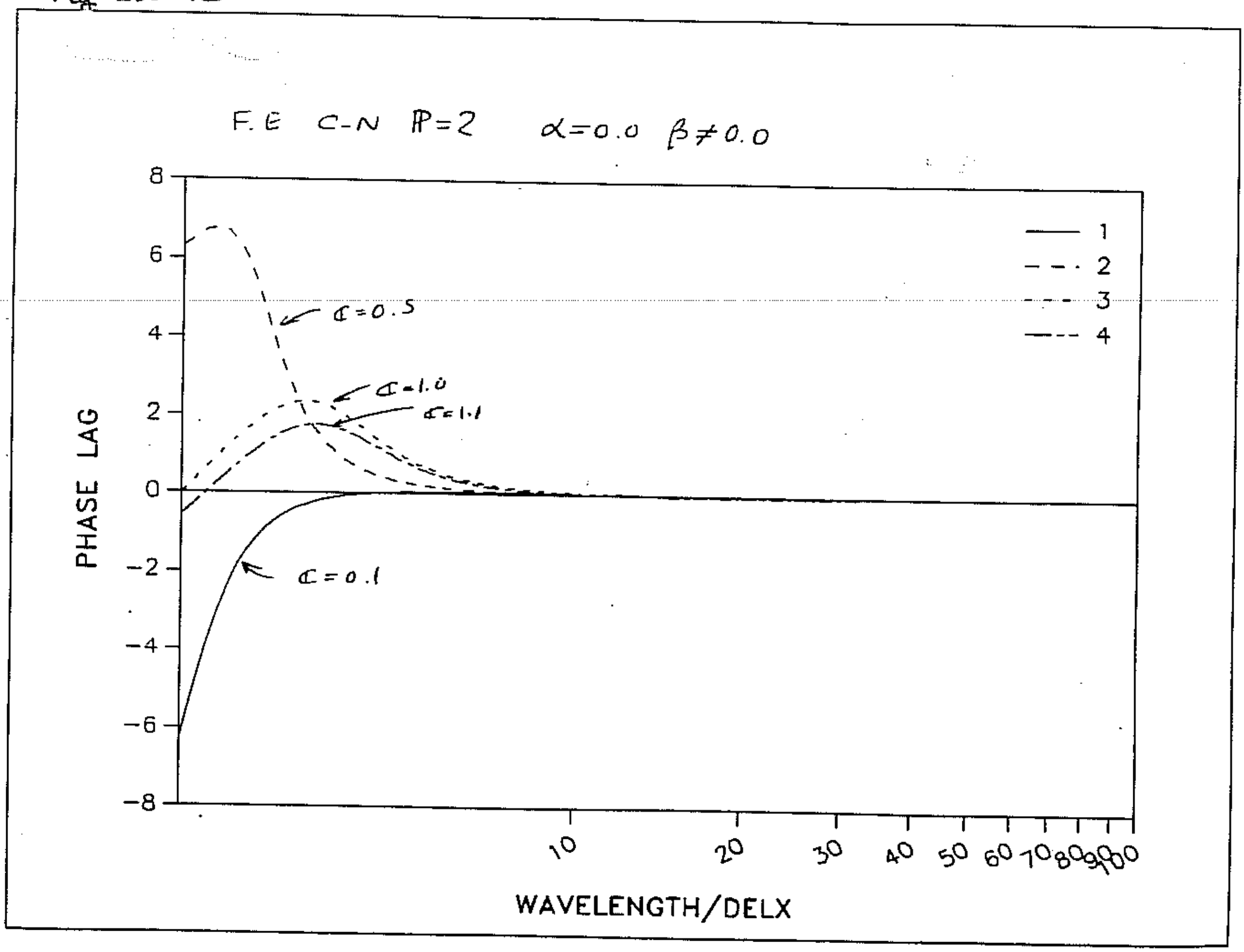


Fig L30.8a

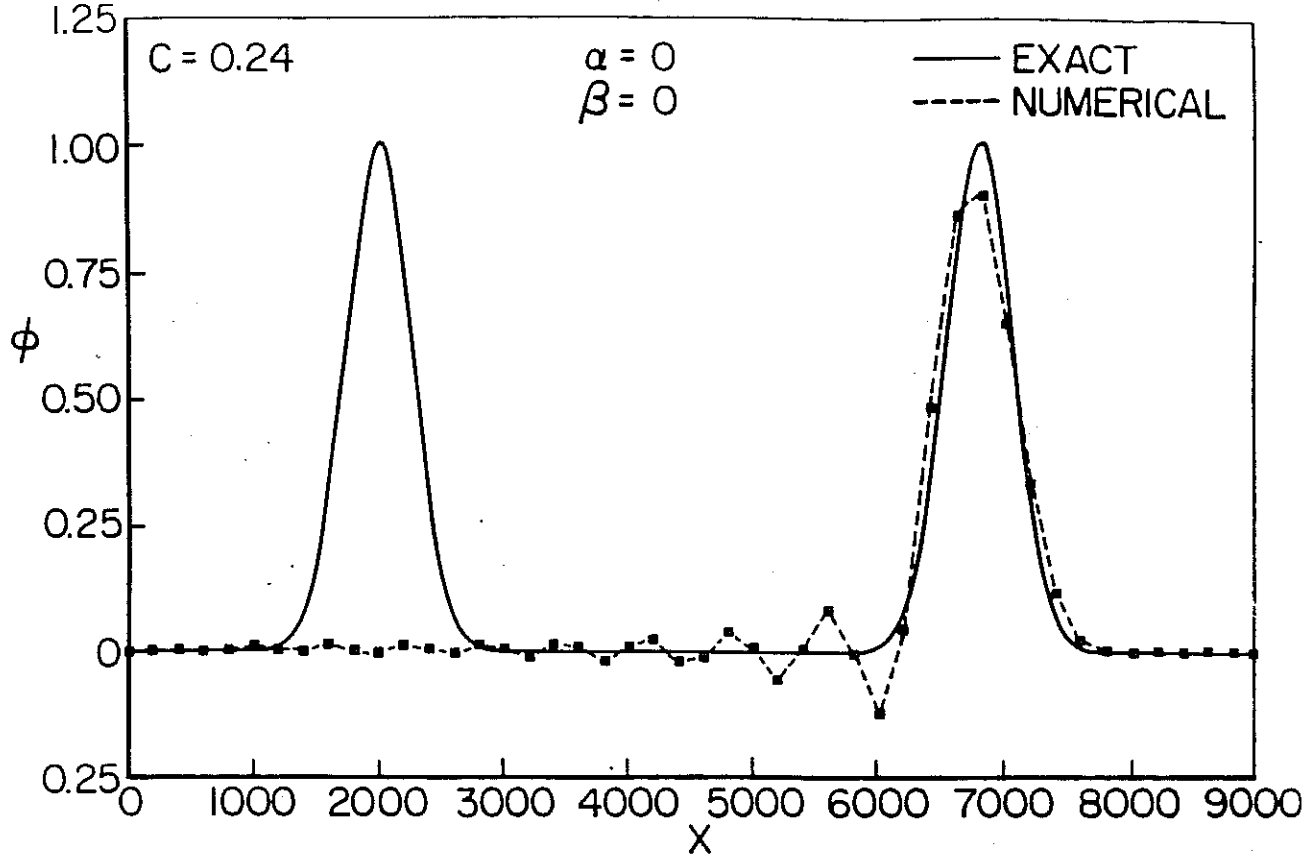


Fig L30.8b

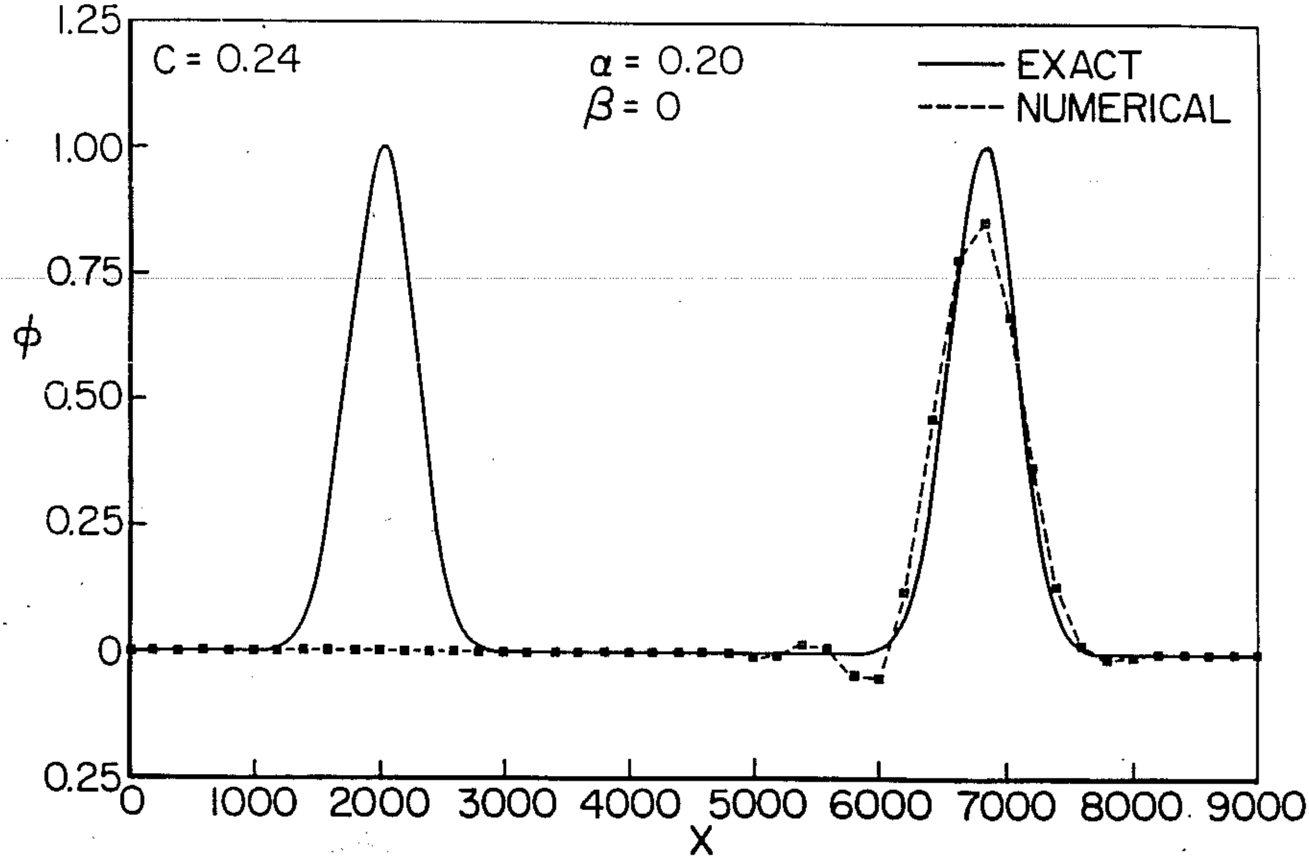


Fig L30.8c

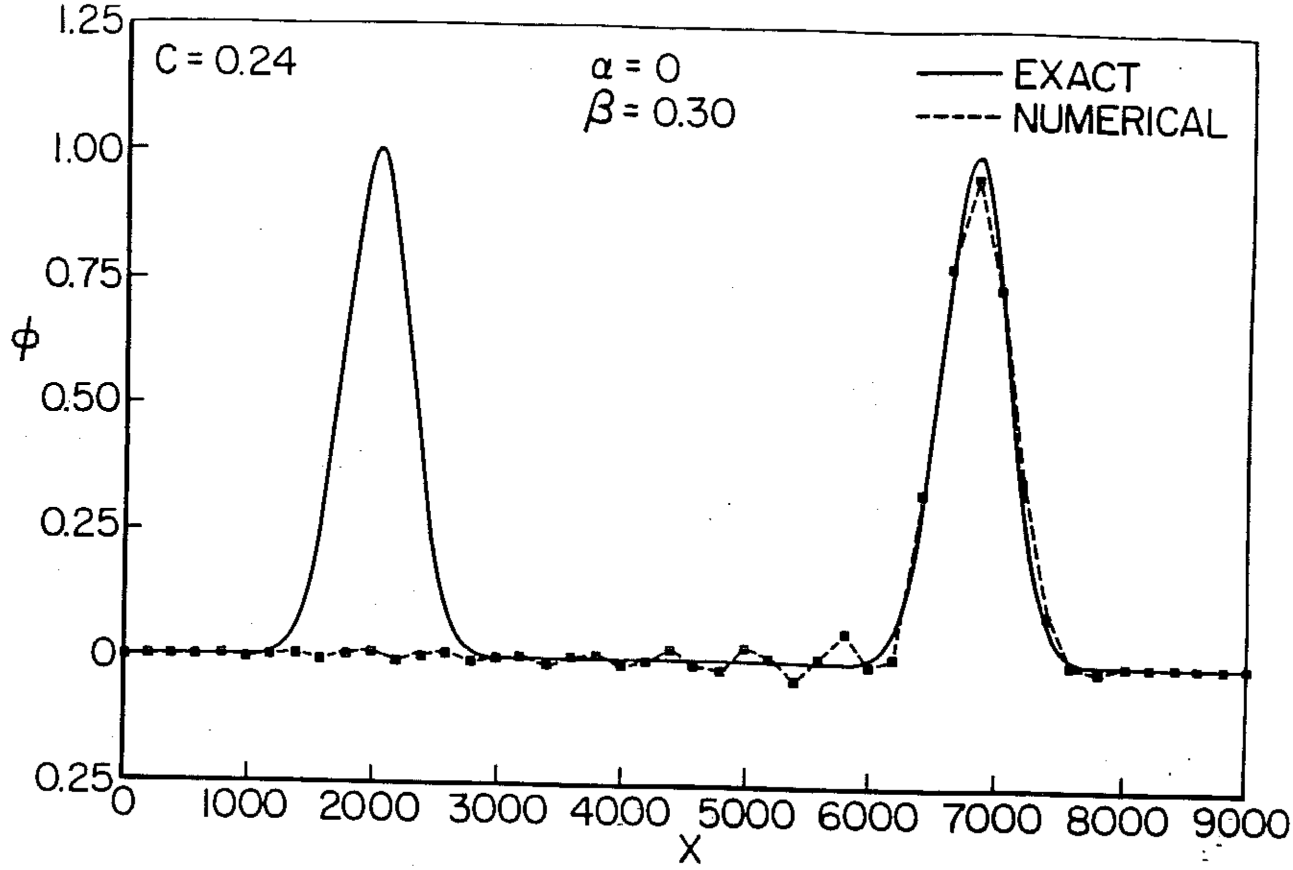


Fig L30.9a

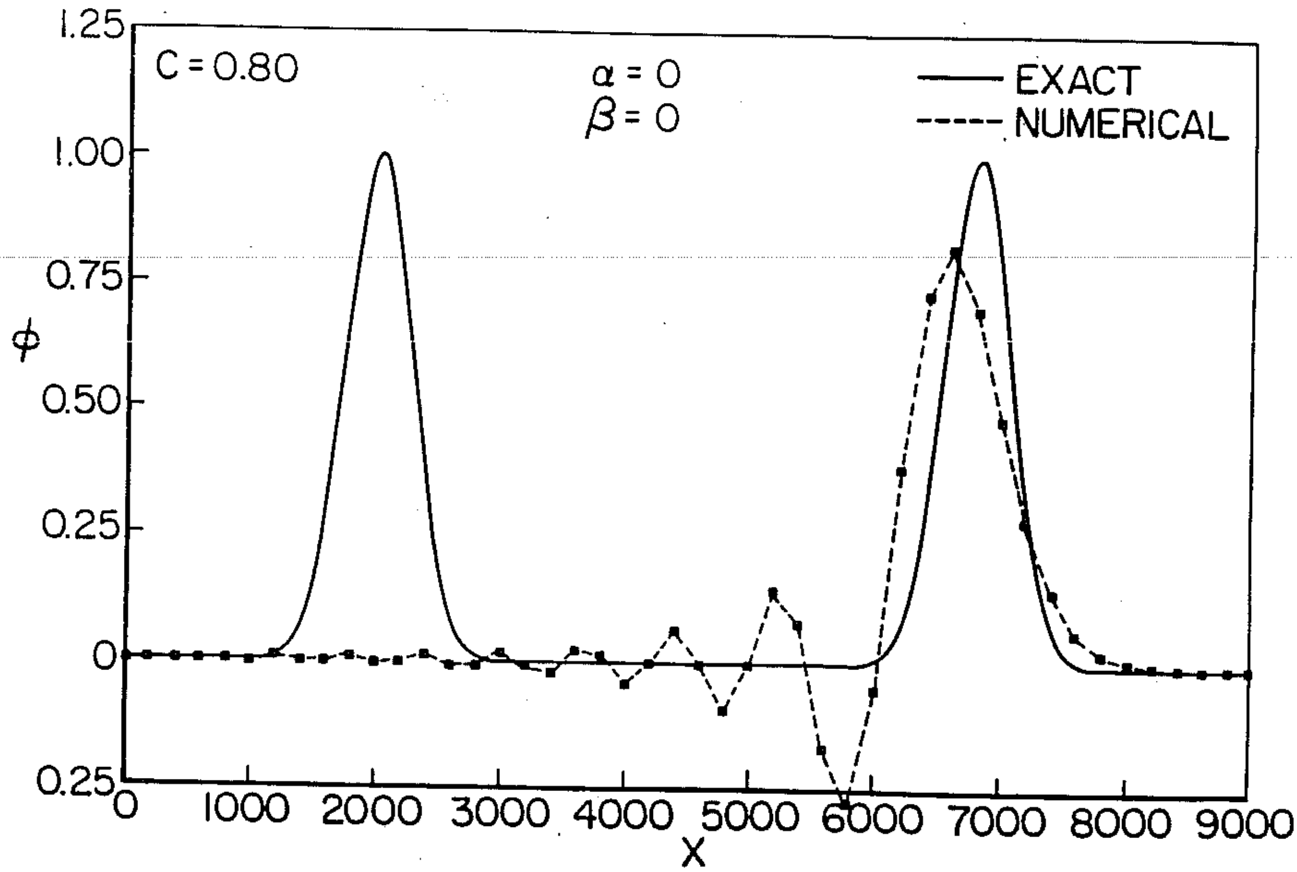


Fig L30.9b

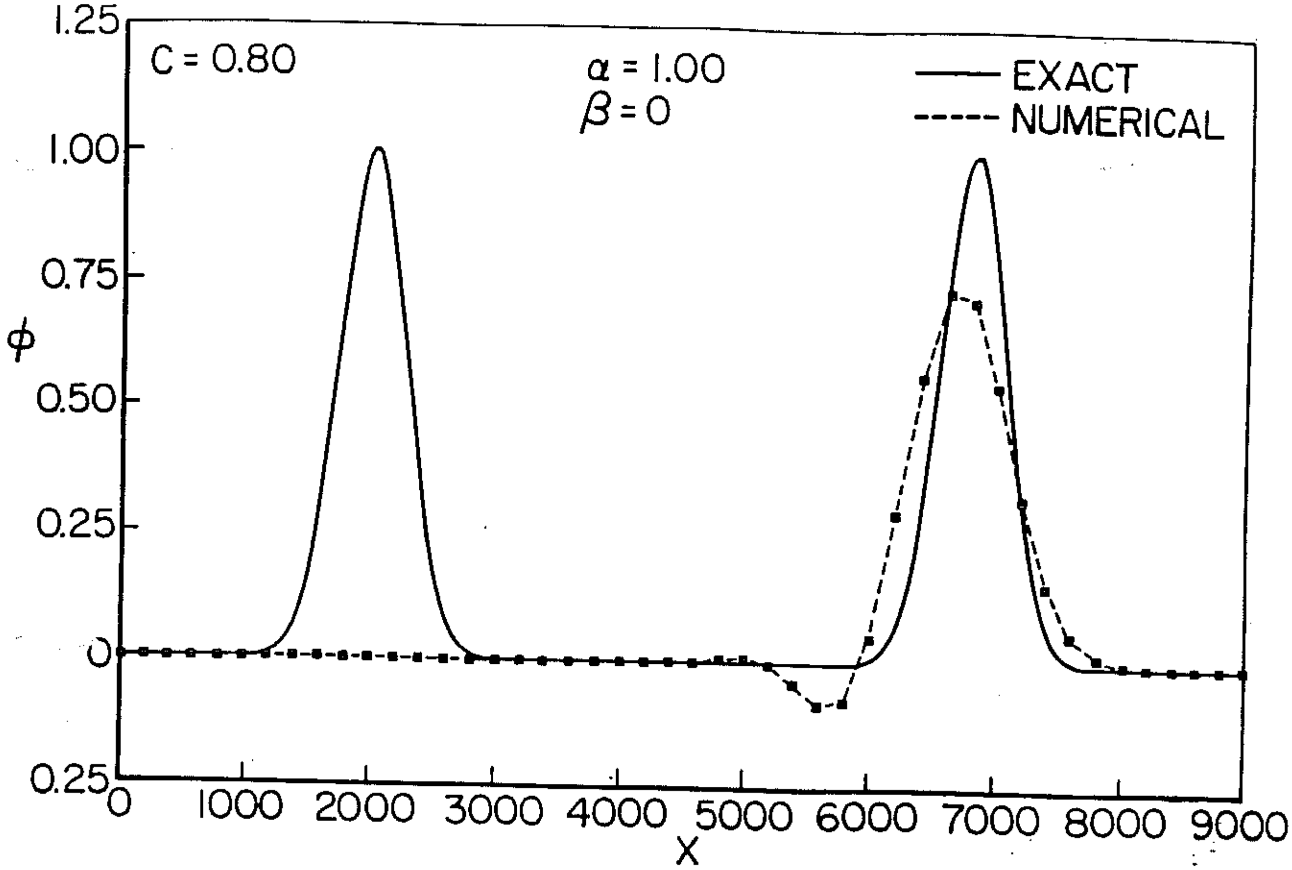


Fig L30.9c

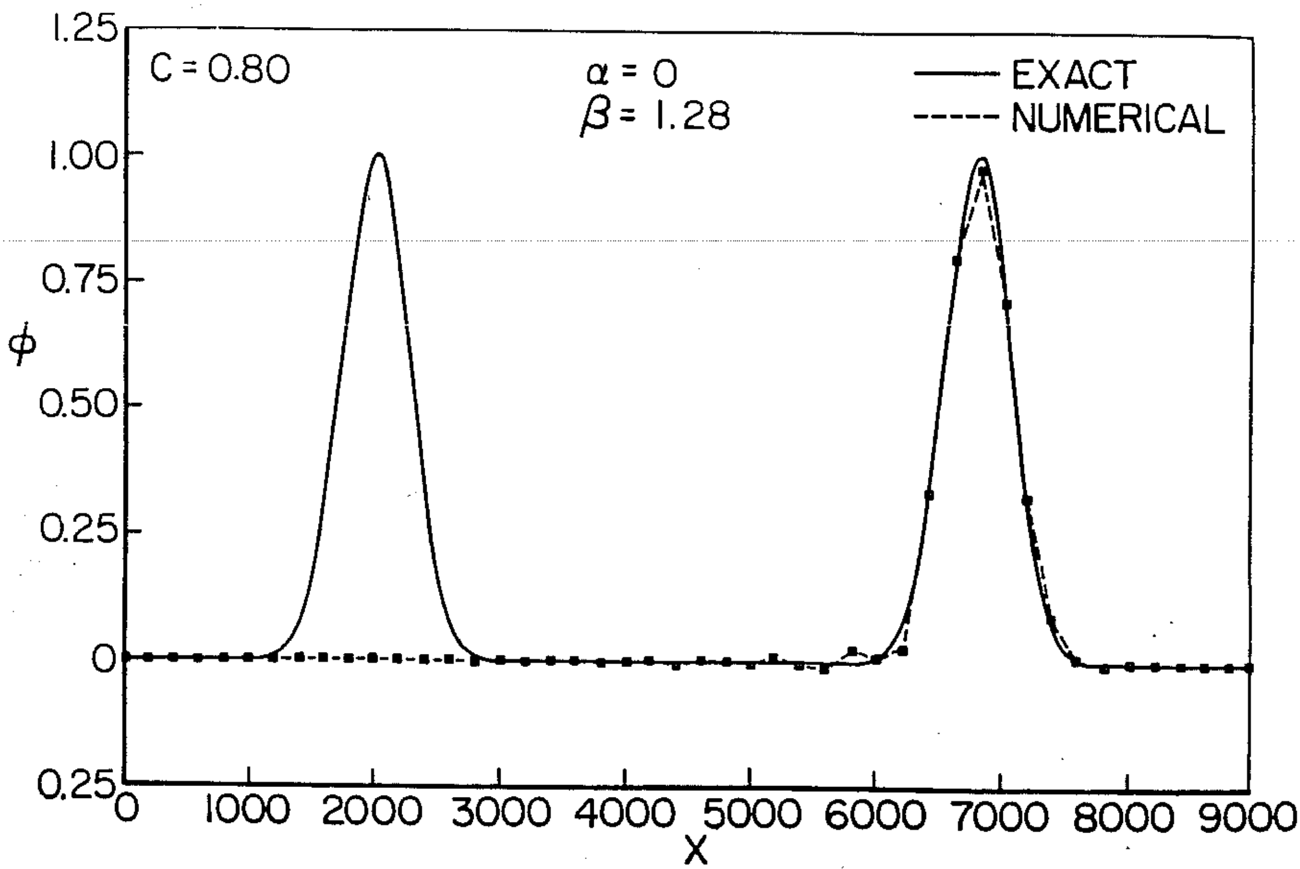


Fig L30.10a

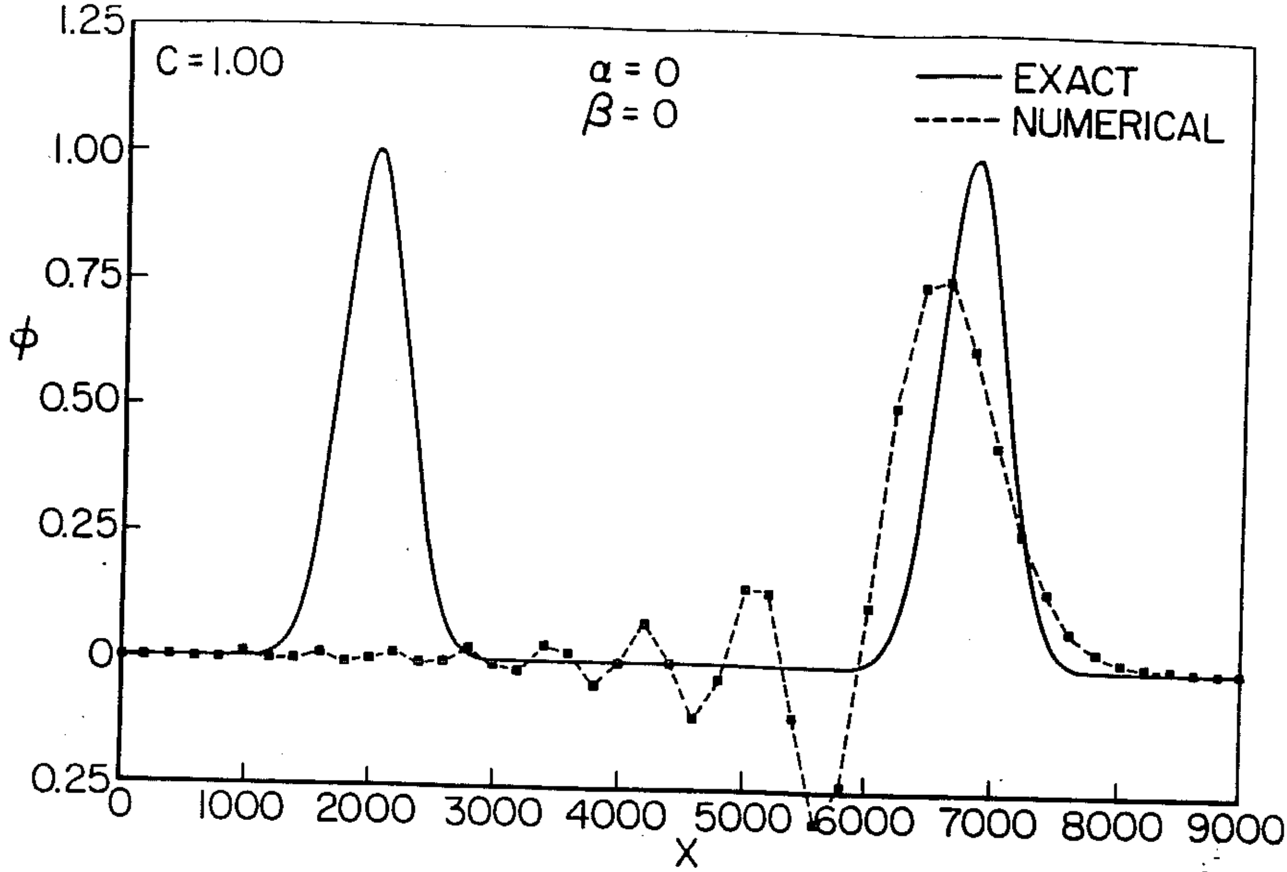


Fig L30.10b

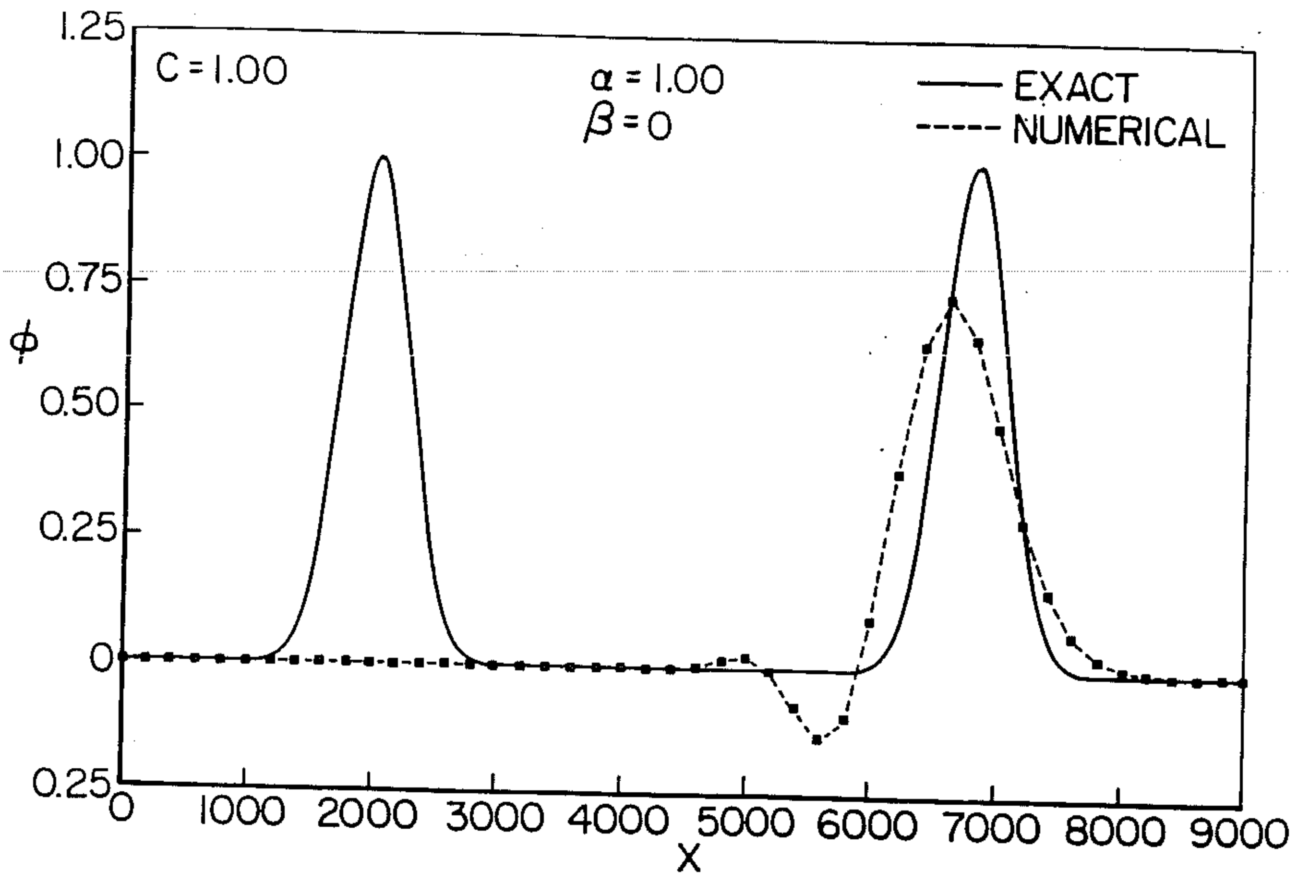


Fig L30.10c

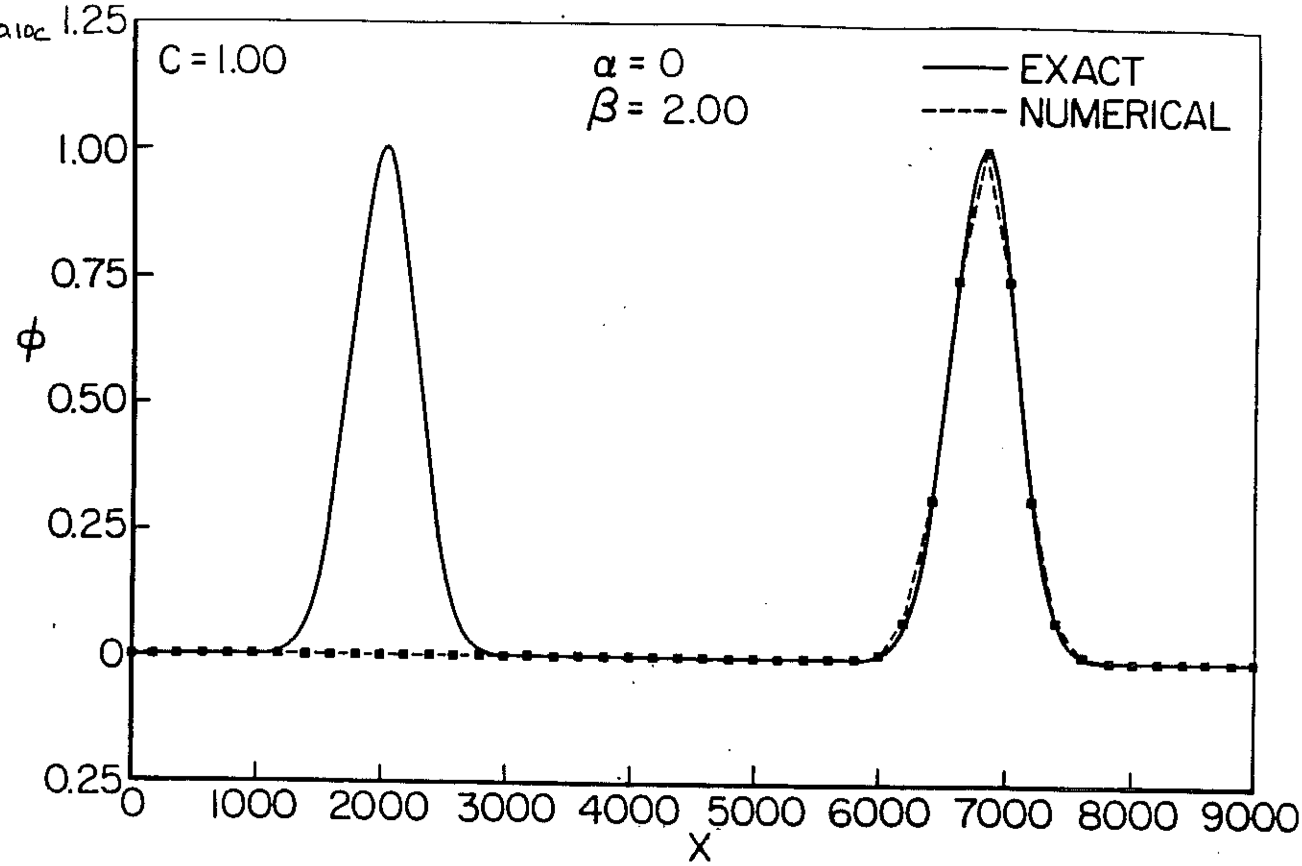


Fig L30.10d

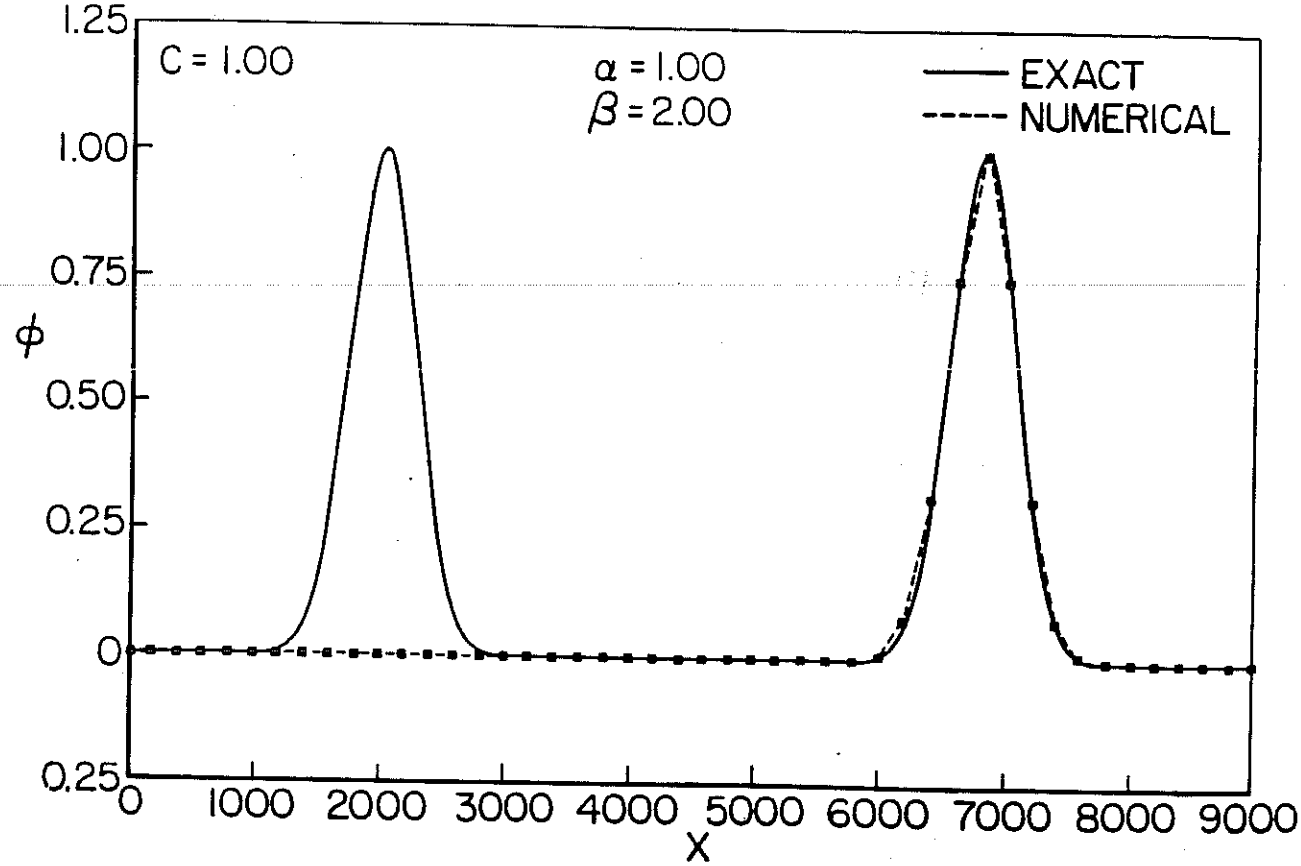


Fig L30.11a

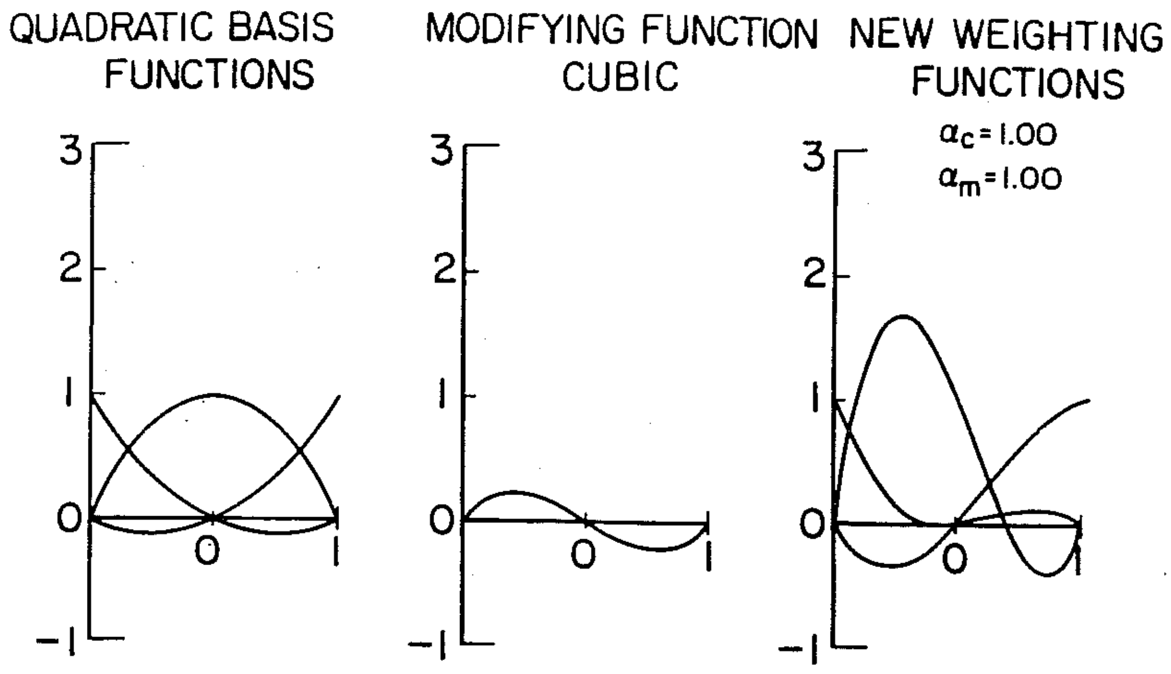
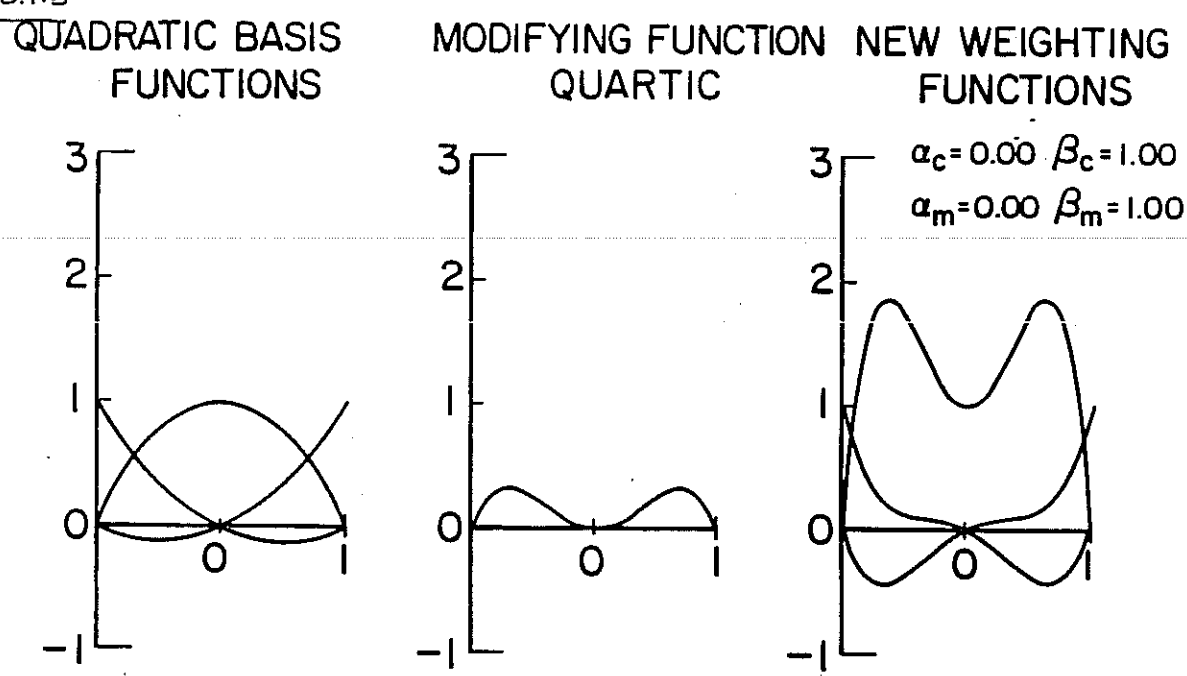


Fig L30.11b



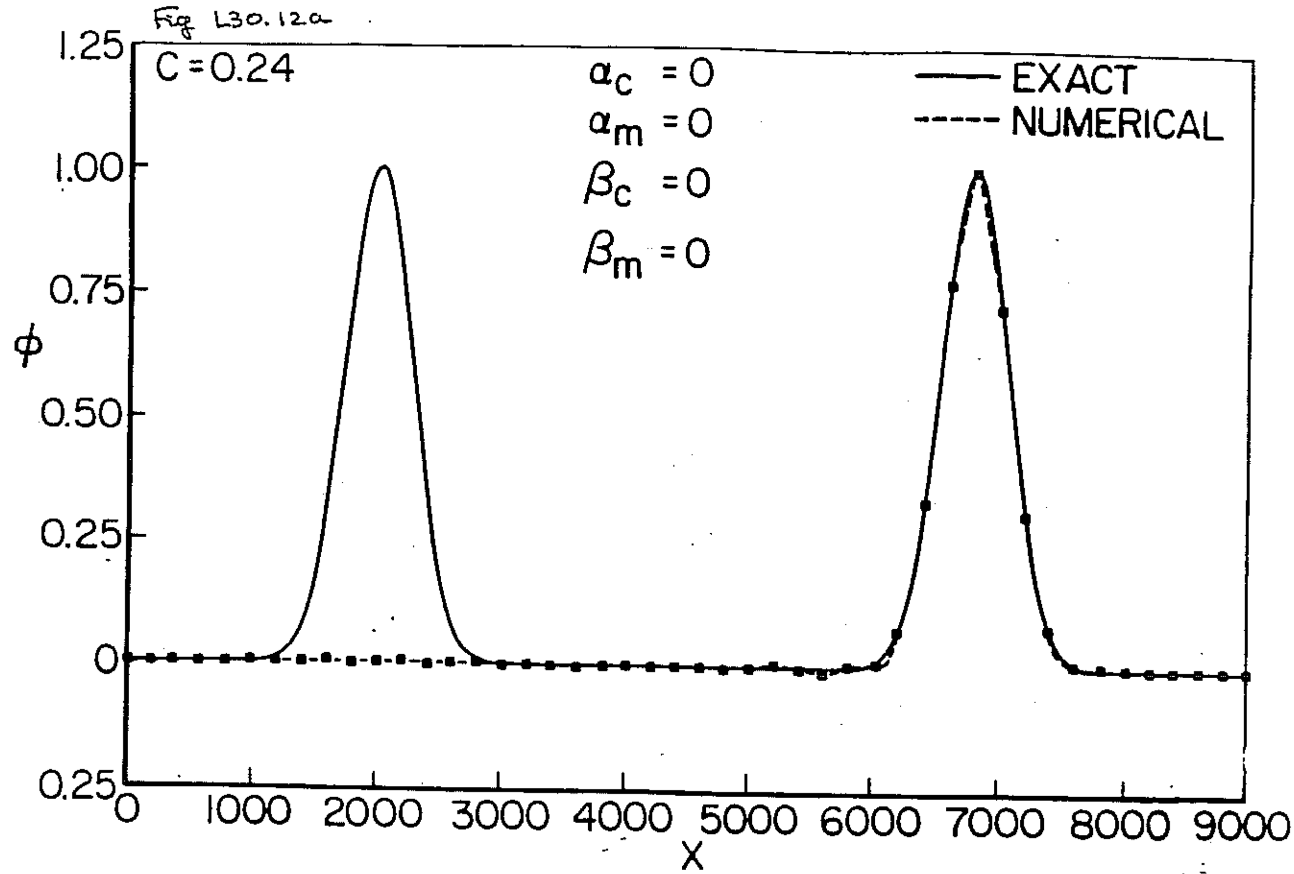


Fig L30.12a
C = 0.24
α_c = 0
α_m = 0
β_c = 0
β_m = 0
EXACT
NUMERICAL
ϕ
X
1.25
1.00
0.75
0.50
0.25
0
0.25
0 1000 2000 3000 4000 5000 6000 7000 8000 9000

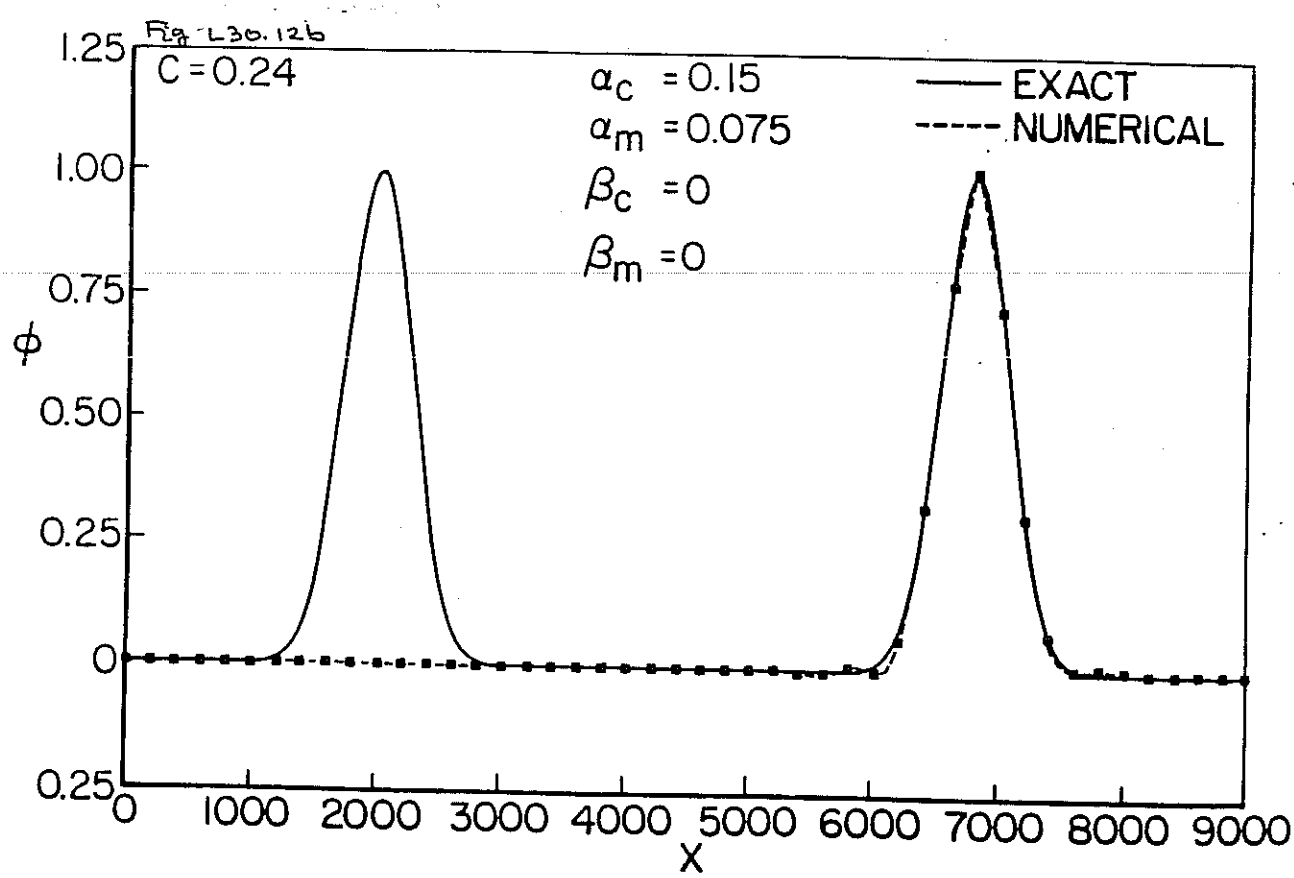


Fig L30.12b
C = 0.24
α_c = 0.15
α_m = 0.075
β_c = 0
β_m = 0
EXACT
NUMERICAL
ϕ
X
1.25
1.00
0.75
0.50
0.25
0
0.25
0 1000 2000 3000 4000 5000 6000 7000 8000 9000

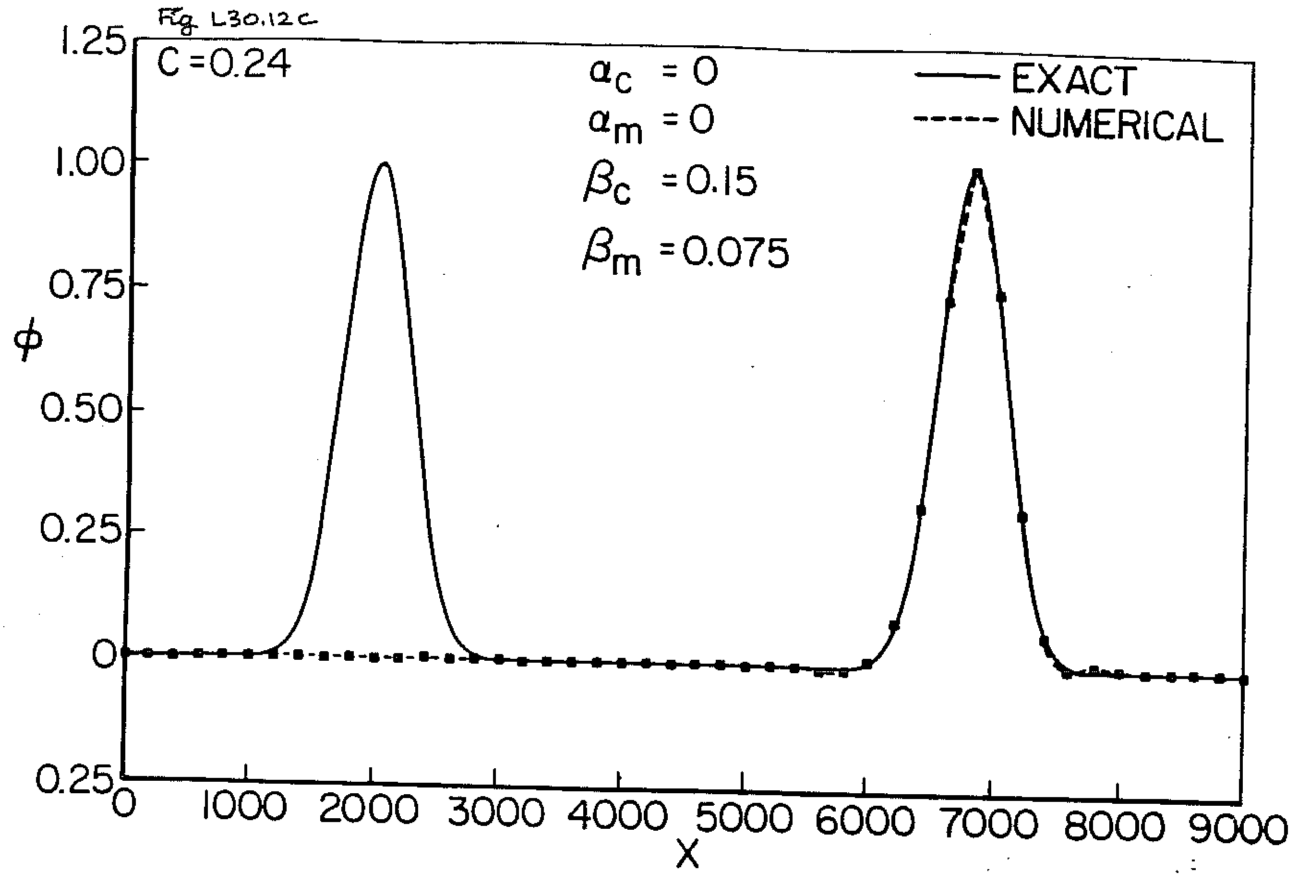


Fig. L30.12c
C = 0.24
α_c = 0
α_m = 0
β_c = 0.15
β_m = 0.075
EXACT
NUMERICAL
1.25
1.00
0.75
0.50
0.25
0
0.25
ϕ
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
X

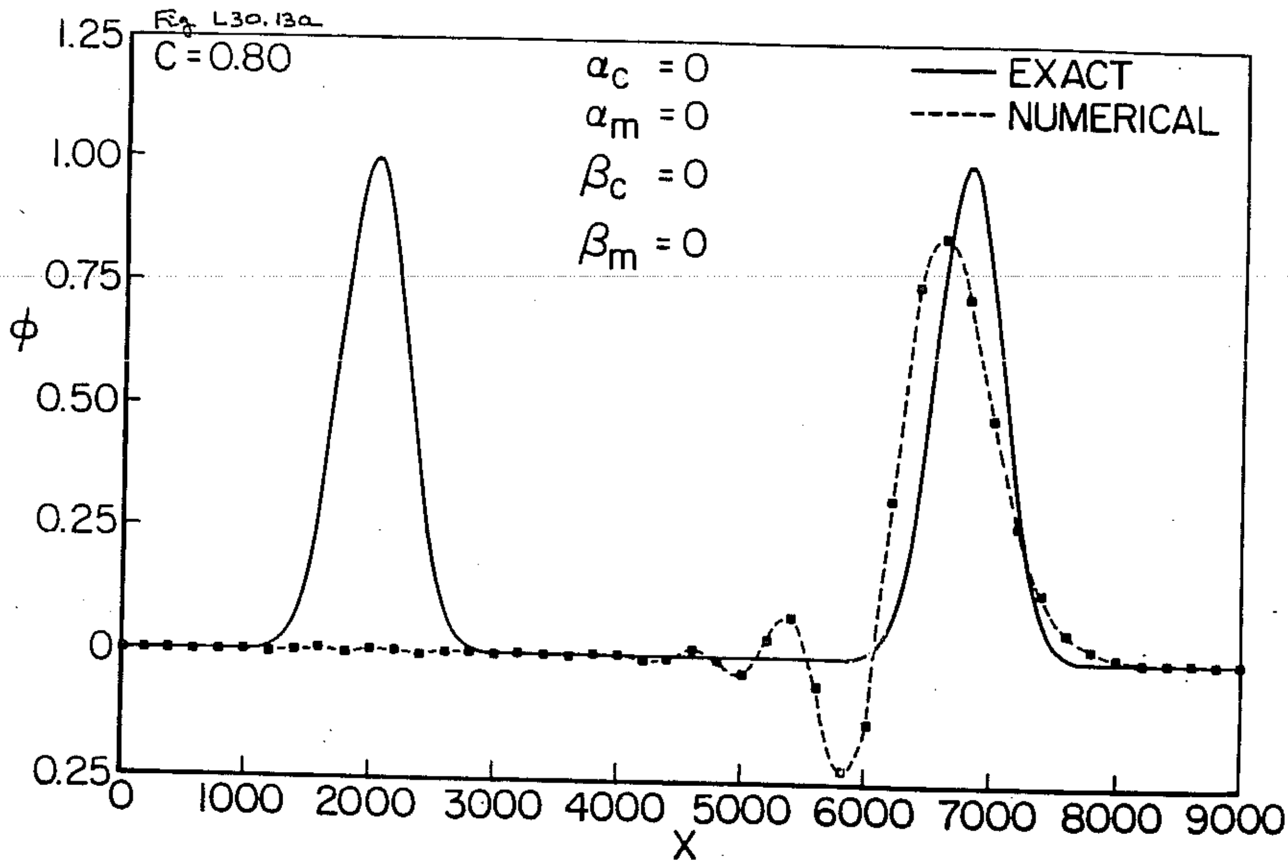


Fig. L30.13a
C = 0.80
α_c = 0
α_m = 0
β_c = 0
β_m = 0
EXACT
NUMERICAL
1.25
1.00
0.75
0.50
0.25
0
0.25
ϕ
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
X

Fig. L30.13b

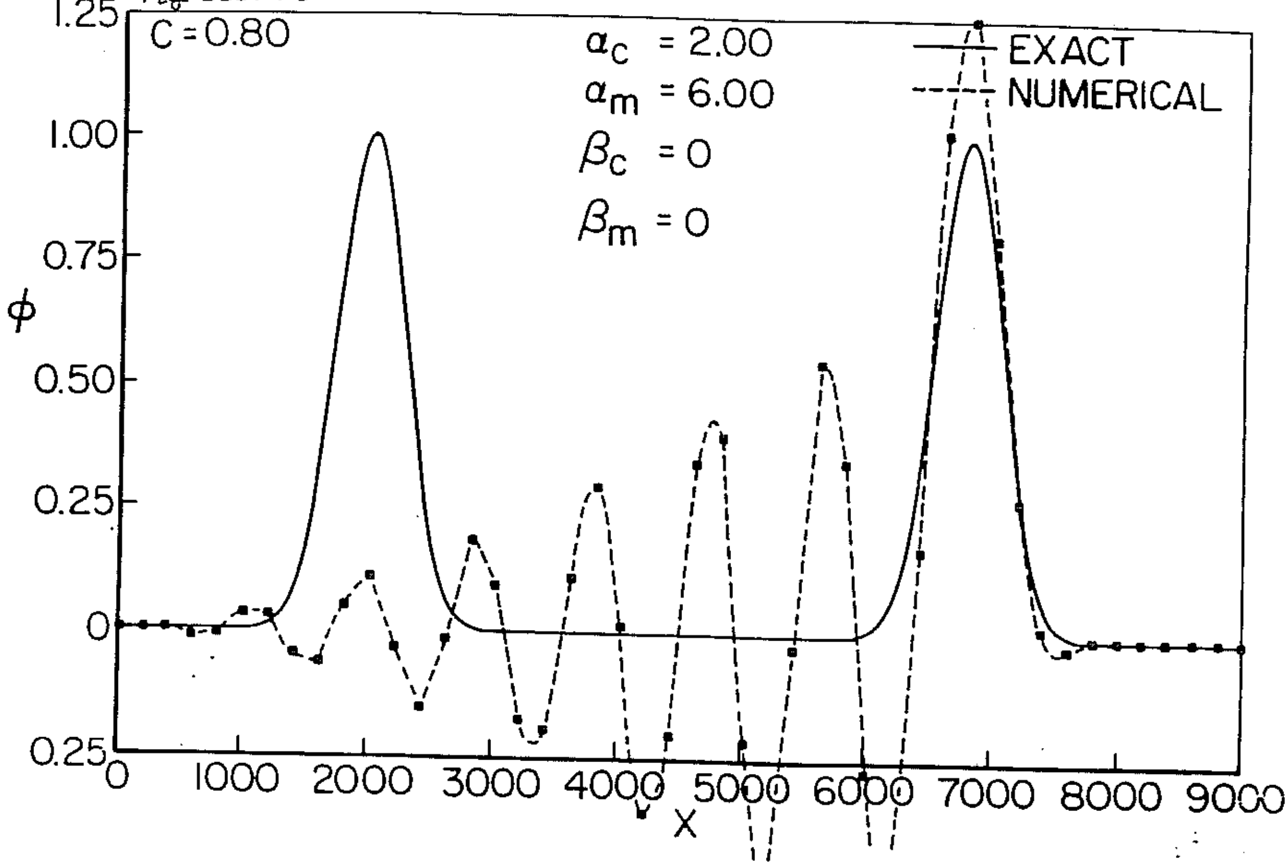


Fig. L30.13c