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Mathematical Methods II

Sectionally Analytic Functions-Plemelj Formulas

Definition: Let C be a closed contour in the plane of the complex variable z. The domain within the contour C is called the interior domain and is denoted D_+. The complementary domain to D_+ and C is called the exterior domain and is denoted D_-.

Theorem : If $f(z)$ is analytic in D_+ and continuous in D_+ and C, then

$$\frac{1}{2\pi i}\int_C \frac{f(\tau)}{\tau - z}d\tau = \begin{cases} f(z) & \text{if } z \in D_+ \\ \frac{1}{2}f(z) & \text{if } z \in C \\ 0 & \text{if } z \in D_- \end{cases}$$

If, however, $f(z)$ is analytic in D_- and continuous in D_- and C, then

$$\frac{1}{2\pi i}\int_C \frac{f(\tau)}{\tau - z}d\tau = \begin{cases} f(\infty) & \text{if } z \in D_+ \\ f(\infty) - \frac{1}{2}f(z) & \text{if } z \in C \\ f(\infty) - f(z) & \text{if } z \in D_- \end{cases}$$

The integration along C is carried in the positive direction(counterclockwise). Note that when $z \in C$, the integral exists only in the sense of the Cauchy principal value.

Theorem : If $\varphi(t) \in \mathcal{H}$ along C, then the function $\Phi(z)$ defined by the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(\tau)}{\tau - z}d\tau$$

is analytic in the complex plane except along C where it has a discontinuity

$$\Phi_+(t) - \Phi_-(t) = \varphi(t)$$

and,

$$\Phi_+(t) + \Phi_-(t) = \frac{1}{\pi i}\int_C \frac{\varphi(\tau)}{\tau - t}d\tau$$

where $\Phi_+(t)$ is the limit of $\Phi(z)$ as $z \to t$ along C while remaining in D_+, and and $\Phi_-(t)$ is the limit of $\Phi(z)$ as $z \to t$ along C while remaining in D_-. Thus, we have

$$\Phi_{\pm}(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\int_C \frac{\varphi(\tau)}{\tau - t}d\tau$$

The integrals are, of course, defined as Cauchy principal values.

Theorem : The functional equation

$$\Phi_{+}(t) - \Phi_{-}(t) = \varphi(\tau)$$

has the particular solution which vanishes at ∞

$$\Phi(z) = \frac{1}{2\pi i}\int_C \frac{\varphi(\tau)}{\tau - z}d\tau$$