Professor H. M. Atassi

CLASS NOTES
ON
QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Definition 1: An equation containing partial derivatives of the unknown function u is said to be an n-th order equation if it contains at least one n-th order derivative, but contains no derivative of order higher than n.

Definition 2: A partial differential equation is said to be linear if it is linear with respect to the unknown function and its derivatives that appear in it.

Definition 3: A partial differential equation is said to be quasilinear if it is linear with respect to all the highest order derivatives of the unknown function.

Example 1: The equation

$$\frac{\partial^2 u}{\partial x^2} + a(x,y)\frac{\partial^2 u}{\partial y^2} - 2u = 0$$

is a second order linear partial differential equation. However, the following equation

$$\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + u^2 = 0$$

is a second order quasilinear partial differential equation. Finally, the equation

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 - u = 0$$

is a first order partial differential equation which is neither linear nor quasilinear.

Definition 4: A solution of a partial differential equation is any function that, when substituted for the unknown function in the equation, reduces the equation to an identity in the unknown variables.

Example 2: Let us consider the one dimensional wave equation

$$(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u = 0.$$

It is well known, and will be shown later, that the general solution of this equation can be cast as

$$u = f(x - ct) + g(x + ct),$$

where f and g are arbitrary twice differentiable functions of the single variables $\xi = x - ct$ and $\eta = x + ct$, respectively. It is easy to see, using the chain rule, that

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{d^2 f}{dx^2} + \frac{d^2 g}{dy^2}),$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dx^2} + \frac{d^2 g}{dy^2}.$$

Substitution into the wave equation leads to an identity.

Definition 5: Let $\phi(\vec{x})$ be a function of the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ having first order derivatives. Then the vector in R^n

$$(D_1, D_2, \ldots, D_n)\phi,$$

where $D_i = \frac{\partial}{\partial x_i}$, is called the gradient of ϕ and usually denoted as $'grad\ \phi'$ or $'\nabla\phi'$.

Definition 6: Let $\vec{v}$ be a unit vector in R^n and v measures the distance on $\vec{v}$. Then the limit

$$\frac{d\phi(\vec{x})}{dv} = \lim_{\Delta t \to 0} \frac{\phi(\vec{x} + \vec{v}\Delta t) - \phi(\vec{x})}{\Delta t},$$

if it exists, is called the derivative of ϕ in the $\vec{v}$ direction. It is easy to show that

$$\frac{d\phi(\vec{x})}{dv} = \nabla\ \phi(\vec{x}) \cdot \vec{v} = (v_1 D_1, v_2 D_2, \ldots, v_n D_n)\phi(\vec{x}),$$

where $\vec{v} = (v_1, v_2, \ldots, v_n)$.

Example 3:
(i) In R^2, $\vec{x} = (x, y)$. Then

$$grad\ \phi(x, y) = (D_x, D_y)\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}).$$

The derivative of ϕ in the direction $\vec{v} = (1/\sqrt{2}, 1/\sqrt{2})$, the first bisector of the plane x-y, is

$$\frac{d\phi}{dv} = \frac{1}{\sqrt{2}}(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}).$$

(ii) Let $\phi(x, y, z)$ be a function having continuous first order derivatives. The equation

$$\phi(x, y, z) = c,$$

where $c \in\ R$, represents a surface S. As we move along a path $\gamma(s)$ on the surface S,

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = 0,$$

$$\frac{d\phi}{ds} = grad\ \phi \cdot (\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}) = 0.$$

For an infinitesimal change in ds, the vector $(dx/ds, dy/ds, dz/ds)$ is in the plane tangent to the surface, at the point $M(x, y, z)$. Consequently, $grad\ \phi$ is orthogonal to the surface $\phi = c$.

2. FIRST ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

We restrict our exposition to first order quasilinear partial differential equations (FOQPDE) with two variables, since this case affords a real geometric interpretation. However, the treatment can be extended without difficulty to higher order spaces.
The general form of FOQPDE with two independent variables is

$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u) \tag{2.1}$$

where a, b, and c are continuous functions with respect to the three variables x, y, u. Let $u = u(x, y)$ be a solution to equation (2.1). If we identify u with the third coordinate z in R^3, then $u = u(x, y)$ represents a surface S. The direction of the normal to S is the vector $(u_x, u_y, -1)$, where u_x, u_y are short hand notation of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, respectively. On the other hand, equation (2.1) can be written as the inner product

$$(u_x, u_y, -1) \cdot (a, b, c) = 0. \tag{2.2}$$

Thus, (a, b, c) is perpendicular to the normal to S and consequently, must lie in the plane tangent to S.
Let us consider a path $\gamma(s)$ on S. The rate of variation of u as we move along γ is

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}. \tag{2.3}$$

The vector $(dx/ds, dy/ds, du/ds)$ is naturally the tangent to the curve $\gamma(s)$. Equation (2.3) can be rewritten as the inner product

$$(u_x, u_y, -1) \cdot (dx/ds, dy/ds, du/ds) = 0. \tag{2.4}$$

Comparing (2.2) and (2.4), we see that there is a particular family of curves on the surface S, defined by

$$\frac{dx/ds}{a} = \frac{dy/ds}{b} = \frac{du/ds}{c}. \tag{2.5}$$

These curves are called characteristics and will be denoted by $C(s)$, or simply C. We note that there are actually only two independent equations in the system (2.5), therefore, its solutions comprise in all a two-parameter family of curves in space.

Theorem 1: Any one parameter subset of the characteristics generates a solution of the first order quasilinear partial differential equation (2.1).
Proof: Let $u = u(x, y)$ be the surface generated by a one-parameter family $C(s)$ of the characteristics whose differential equations are (2.5). By taking the rate of variation of u along a characteristic curve $C(s)$, we get

$$\frac{du}{ds} = u_x\frac{dx}{ds} + u_y\frac{dy}{ds},$$

or

$$(u_x, u_y, -1) \cdot (dx/ds, dy/ds, du/ds) = 0. \tag{2.6}$$

But equations (2.5) state that the vectors $(dx/ds, dy/ds, du/ds)$ and (a, b, c) are collinear. Therefore,

$$(u_x, u_y, -1) \cdot (a, b, c) = 0,$$

or

$$au_x + bu_y = c,$$

and we are done.

Corollary 1: The general solution to equation (2.1) is defined by a single relation between two arbitrary constants occurring in the general solution of the system of ordinary differential equations

$$\frac{(dx/ds)}{a} = \frac{(dy/ds)}{b} = \frac{(du/ds)}{c};$$

or, in other words, by any arbitrary function of one independent variable.

Example 1: Consider the first order partial differential equation,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

The characteristics are defined by

$$\frac{(dt/ds)}{1} = \frac{(dx/ds)}{u} = \frac{(du/ds)}{0}.$$

The last equation gives immediately $u = k_1$. Since u is constant along a given characteristic, then the first equation can be integrated immediately:

$$x - k_1 t = k_2$$

The general solution is then

$$k_1 = f(k_2),$$

where f is an arbitrary function of the single variable k_2. Substituting k_1 and k_2 by their expressions, we finally have

$$u = f(x - ut).$$

Example 2: Consider the equation

$$3\frac{\partial u}{\partial x} - 7\frac{\partial u}{\partial y} = 0.$$

The characteristics are solution of the system

$$\frac{(dx/ds)}{3} = \frac{(dy/ds)}{-7} = \frac{(du/ds)}{0}.$$

By integration, we get

$$u = k_1,$$

$$3y + 7x = k_2.$$

The characteristics are straight lines intersection of the two families of planes defined by these equations. Any arbitrary relation between k_1 and k_2 is a solution. The general solution is then

$$u = f(3y + 7x),$$

where f is an arbitrary function of the one independent variable $3y + 7x$.

Example 3: Consider the equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

The characteristics are solution of the system

$$\frac{(dx/ds)}{y} = \frac{(dy/ds)}{-x} = \frac{(du/ds)}{0}.$$

By integration, we get

$$u = k_1,$$

$$x^2 + y^2 = k_2.$$

The characteristics are circles located in the plane $u = k_1$. The general solution is

$$u = f(x^2 + y^2),$$

where f is an arbitrary function of the independent variable $(x^2 + y^2)$. Geometrically, the general solution is any surface of revolution around the u-axis.

Remark 1: The previous examples illustrate the application of the theory of characteristics to find the general solution to a first order quasilinear partial differential equation. Given the simple forms of these examples, the student could become suspicious that in the general case it will not be possible to get a closed analytical form of the solution. A careful examination of system (2.5) suffices to convince you that, in the general case, i.e., when a, b, and c are arbitrary functions of x, y, and u, the system of ordinary differential equations (2.5) cannot be integrated analytically. Nevertheless, for a given initial or boundary value problem, system (2.5) provides a numerical solution.

Remark 2: When equation (2.1) is a linear homogeneous partial differential equation

$$a(x, y)u_x + b(x, y)u_y = c(x, y)u. \tag{2.7}$$

The solutions form a vector space which we call the solution space. This solution space is naturally the null space of the linear operator

$$a(x, y)D_x + b(x, y)D_y - c(x, y). \tag{2.8}$$

It should be pointed out that since the general solution to (2.7) is any arbitrary function of one independent variable, there is a nondenumerable infinity of independent solutions to equations (2.7). Therefore, the dimension of null space of the operator (2.8) is infinity. This result appears to be in sharp contrast with what we know about the first order ordinary linear differential operators where the null space is one-dimensional. One may also add that this augurs the difficulties we shall encounter in the study of partial differential operators.

THE BOUNDARY VALUE PROBLEM FOR A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION

The theory of characteristics enables us to define the solution to FOQPDE (2.1) as surfaces generated by the characteristic curves defined by the ordinary differential equations (2.5). However, a physical problem is not uniquely specified if we simply give the differential equation which the solution must satisfy, for, as we have seen, there are an infinite number of solutions of every equation. In order to make the problem a definite one, with a unique answer, we must pick out of the mass of possible solutions, the one which has certain definite properties along definite boundary surfaces. These properties represent the boundary conditions which the solution must satisfy. The first fact which we must notice is that we cannot try to make the solutions of a given equation satisfy any sort of boundary conditions; for there is a definite set of boundary conditions which will give nonunique or impossible answers. The study of the proper boundary conditions to be specified on definite boundary curves or surfaces is often termed the Cauchy problem, in honor of the French mathematician who laid the foundations to our present knowledge in this area.

Statement of the Cauchy problem for FOQPDE

Let $\gamma(t)$ be a curve in region R of the x-y plane, where the value of the function $u(x, y)$ which satisfies equation (2.1) is specified as $u(t)$. What are the conditions to be satisfied by $\gamma(t)$ and $u(t)$, in order that the boundary value problem so defined has a unique solution?
Consider an initial curve $\gamma(t)$ in a region R

$$\begin{aligned} \xi &= \xi(t), \\ \eta &= \eta(t). \end{aligned} \tag{2.9}$$

Since $u(x, y)$ is specified on $\gamma(t)$ as

$$u(x, y) = u(t) \tag{2.10}$$

then the three functions

$$\xi = \xi(t),\ \eta = \eta(t),\ u = u(t) \tag{2.11}$$

define a curve Γ in space whose projection over the x-y plane is γ. By a proper choice of the parameter s, the equations of the characteristics are

$$\frac{dx}{ds} = a, \ \frac{dy}{ds} = b, \ \frac{du}{ds} = c, \tag{2.12}$$

The characteristic curve passing by a point $M(t) \in \Gamma$ is then defined by the following equations

$$x = x(s,t), \ y = y(s,t), \ u = u(s,t). \tag{2.13}$$

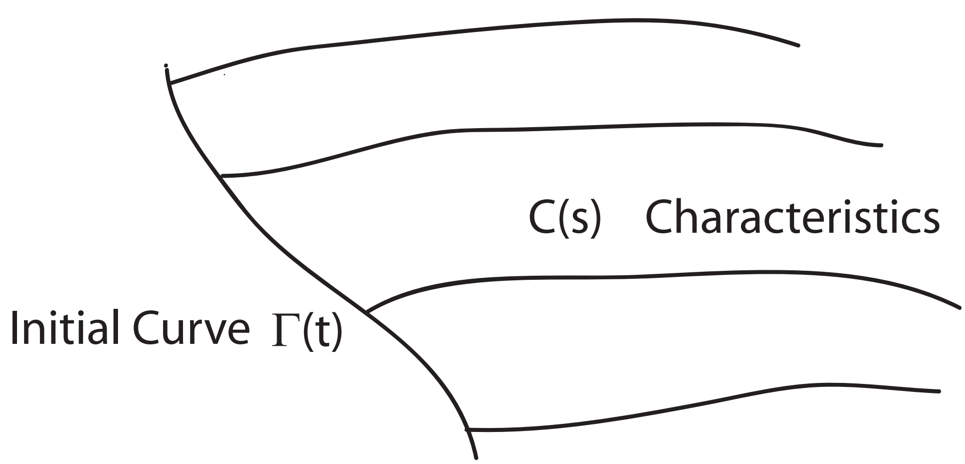

Figure 2.1: Solution of the boundary-value problem of a first-order partial differential equation by a one-parameter family of characteristics.

Equation (2.13) represent the surface S in a parametric form. When t is kept constant we move along a characteristic curve $C(s)$. For a given value of s, for example s_0, we move on the surface S along the curve Γ.
Let us now assume that the initial curve Γ was a characteristic curve C. It is then obvious that equations (2.9) satisfy system (2.12) and there will be no surface solution generated this way. We conclude that in order to generate a surface solution, Γ should not be a characteristic curve. Mathematically, this means that the two vectors $(d\xi/dt, d\eta/dt, du/dt)$ and (a, b, c) must be linearly independent. A sufficient condition to insure this linearly independence is that the determinant

$$J = \begin{vmatrix} \frac{d\xi}{dt} & \frac{d\eta}{dt} \\ a & b \end{vmatrix} \tag{2.14}$$

never vanishes on Γ.
An alternative form to the condition (2.14) can be obtained from equations (2.13). The vectors tangent to C and Γ are respectively (x_s, y_s, u_s) and (x_t, y_t, u_t). A sufficient condition for their independence is that the Jacobian

$$J = \begin{vmatrix} x_s & y_s \\ x_t & y_t \end{vmatrix}$$

never vanishes on Γ. We are then led to the following theorem:

Theorem 2: The solution $u(x, y)$ may be freely specified along a curve γ in R and the resulting specification determines a unique solution of (2.1) if and only if γ intersects the projection on the x-y plane of each characteristic curve exactly once.

Example 4:
Take the simple equation

$$u_x + u_y = c(x, y, u) \tag{2.15}$$

and take R as the square region in Figure 2.2. The projection on the x-y plane of the characteristic curves (often called the characteristics) are the lines $x - y = \lambda$, for $x, y \in R$. A few of these characteristics are shown in Figure 2.2 (a). We see that the curve γ_1 of Figure 2.2 (b) meets the conditions of theorem 2 so that specification of $u(x, y)$ along γ_1 converts (2.1) into a boundary-value problem with a unique solution. However, Curve γ_2 of Figure 2.2(c) while not intersecting any characteristic more than once does not intersect all the characteristics in R, so the specification of $u(x, y)$ along γ_2 would specify $u(x, y) = u(t)$ only for those characteristics which intersect γ_2, i.e., inside a sub region of R between the limiting characteristics shown in Figure 2.2(c), which is called *domain of influence* of the curve γ_2. Specification of $u(x, y)$ along γ_2, then determines v only in the domain of influence of γ_2 and we would say such a boundary-value problem was **understated** for the whole region R. The curve γ_3 of Figure 2.2(d)intersects some characteristics more than once, so specification of $u(x, y)$ along v would convert (2.1) to a boundary-value problem with no solution unless we were lucky enough that the specified values for the several points on the same characteristics were compatible with the general solution; we say, this boundary-value problem is **overspecified**.

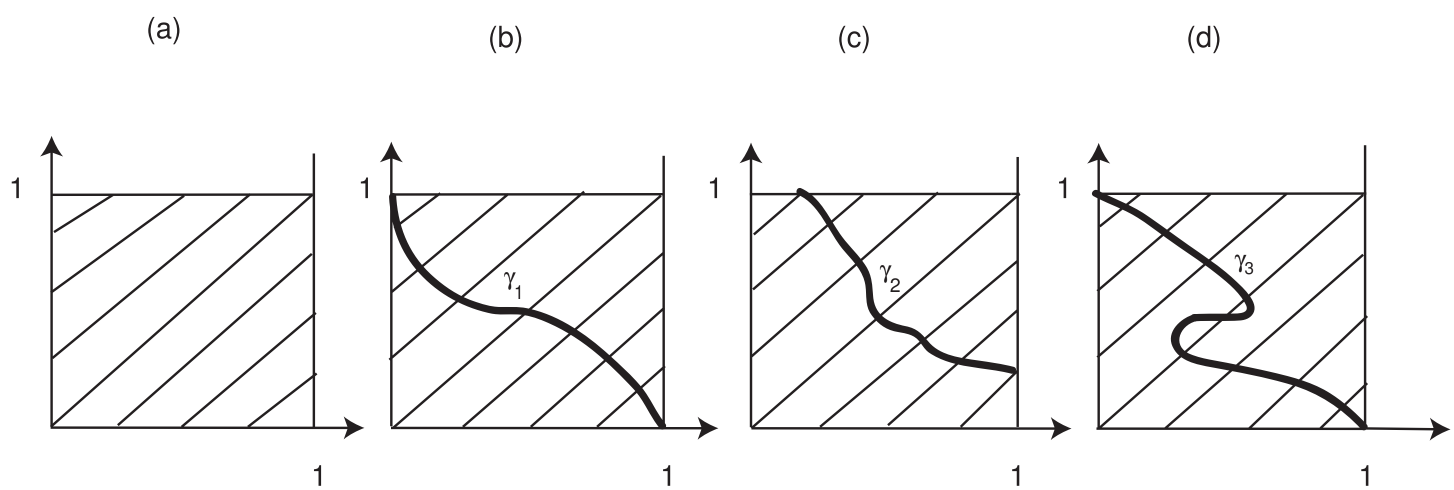

Figure 2.2: (a) Characteristics of of (2.15), (b) Well-posed boundary-value problem, (c) Understated boundary-value problem, (d) Overspecified boundary-value problem.

Theorem 3: The boundary-value problem for FOQPDE

$$au_x + bu_y = c, \; x, \; y, \; in \; R$$

$$u(x,y) = u(t), \; x, \; y, \; on \; \gamma$$

has a unique solution for any specified $u(t)$ if and only if γ intersects the projection of each characteristic curve in R exactly once.

Remark 3: Due to the nonlinearity of (2.1), the domain of influence has only local significance. The geometry of the characteristics can become quite complicated to the point that a well posed boundary-value problem could be very difficult to solve as we move along the characteristics.

3. SYSTEMS OF FIRST ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

We denote by u, v the dependent variables (unknown functions) and x, y the independent variables. The general form of the system of differential equations is

$$L_1 = A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0,$$

$$L_2 = A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0, \tag{3.1}$$

in which A_1, A_2, $\cdots$, E_2 are known functions of x, y, u, v. We make the assumption that all functions occurring in the theory are continuous and possess as many continuous derivatives as may be required. Without restrictions we assume that nowhere

$$A_1/A_2 = B_1/B_2 = C_1/C_2 = D_1/D_2.$$

If $E_1 = E_2 = 0$, the system is homogeneous. If the coefficients A_1, A_2, $\cdots$, E_2 are functions of x and y only, the equations are linear and consequently much easier to handle.

Definition 1: If the system is homogeneous, $E_1 = E_2 = 0$, and the coefficients A_1, A_2, $\cdots$, D_2 are functions of u, v alone, the equations are said to be reducible. The following theorem justifies the above definition.

Theorem 1: For any region where the Jacobian

$$j = u_x v_y - u_y v_x \tag{3.2}$$

is not zero, a reducible system (3.1) can be transformed by interchanging the roles of the dependent and independent variables into the following equivalent linear system

$$A_1 y_v - B_1 x_v - C_1 y_u + D_1 x_u = 0,$$

$$A_2 y_v - B_2 x_v - C_2 y_u + D_2 x_u = 0. \tag{3.3}$$

The transformation of the (x, y)- plane into the (u, v)- plane is called a hodograph transformation. It is advantageous, in general, to perform the hodograph transformation whenever the Jacobian $j \neq 0$ in the region where a solution is desired.

Characteristic curves and characteristic equations

We have shown (Definition 1.6) that an expression such as $A_1 u_x + B_1 u_y$ represents derivative in a direction $dx/dy = A_1/B_1$. Each equation of (3.1) can be thought of as a linear relation between a derivative of u in the direction (A, B) and a derivative of v in the direction (C, D). We now ask for a linear combination

$$L = \lambda_1 L_1 + \lambda_2 L_2,$$

so that in the differential expression L, the derivatives of u and those of v combine to derivatives in the same direction. Such direction, which depends on the variables x, y as well as u, v, is called characteristic direction.
Suppose the direction is given by the ratio x_σ/y_σ. Then the condition that, in L, u and v are differentiated in this direction, is simply

$$\frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2} = \frac{x_\sigma}{y_\sigma} \tag{3.4}$$

The expression L can be written after multiplication with either x_σ or y_σ as

$$\begin{gathered}(\lambda_1 A_1 + \lambda_2 A_2)u_\sigma + (\lambda_1 C_1 + \lambda_2 C_2)v_\sigma + (\lambda_1 E_1 + \lambda_2 E_2)x_\sigma = x_\sigma L \\ \\ (\lambda_1 B_1 + \lambda_2 B_2)u_\sigma + (\lambda_1 D_1 + \lambda_2 D_2)v_\sigma + (\lambda_1 E_1 + \lambda_2 E_2)y_\sigma = y_\sigma L\end{gathered} \tag{3.5}$$

If at the point $(x,\, y)$, the functions u and v satisfy (3.1), we obtain four homogeneous linear equations for λ_1 and λ_2:

$$\begin{aligned} \lambda_1(A_1 y_\sigma - B_1 x_\sigma) + \lambda_2(A_2 y_\sigma - B_2 x_\sigma) &= 0, \\ \lambda_1(C_1 y_\sigma - D_1 x_\sigma) + \lambda_2(C_2 y_\sigma - D_2 x_\sigma) &= 0, \\ \lambda_1(A_1 u_\sigma + C_1 v_\sigma + E_1 x_\sigma) + \lambda_2(A_2 u_\sigma + C_2 v_\sigma + E_2 x_\sigma) &= 0, \\ \lambda_1(B_1 u_\sigma + D_1 v_\sigma + E_1 y_\sigma) + \lambda_2(B_2 u_\sigma + D_2 v_\sigma + E_2 y_\sigma) &= 0. \end{aligned} \tag{3.6}$$

In order to have nontrivial solutions to (3.6), all determinants of two rows in the matrix of the coefficient of λ_1 and λ_2 must vanish. Thus a number of characteristic relations follow. From the first two equations, we get

$$\begin{vmatrix} A_1 y_\sigma - B_1 x_\sigma & A_2 y_\sigma - B_2 x_\sigma \\ C_1 y_\sigma - D_1 x_\sigma & C_2 y_\sigma - D_2 x_\sigma \end{vmatrix} = 0 \tag{3.7}$$

or

$$a y_\sigma^2 - 2b x_\sigma y_\sigma + c x_\sigma^2 = 0, \tag{3.8}$$

where

$$a = [AC],\ 2b = [AD] + [BC],\ c = [BD]$$

with the abbreviation

$$[XY] = X_1Y_2 - X_2Y_1.$$

Equation (3.8) defines the directions of the characteristics. There are three cases:

(i) $ac - b^2 > 0$, then (3.8) cannot be satisfied by a real direction. The differential equations (3.1) are then called elliptic.

(ii) $ac - b^2 = 0$, then there is only one characteristic direction. The differential equations (3.1) are then called parabolic.

(iii) $ac - b^2 < 0$, we have two different characteristic directions through each point. The differential equations (3.1) are then called hyperbolic.

It is obvious that the notion of characteristics is useful only when they do exist. For this reason, we shall from now on assume the hyperbolic character of system (3.1) and accordingly suppose

$$ac - b^2 < 0.$$

Let $\xi = y_\sigma/x_\sigma$ be the slope of the characteristics, then ξ satisfies the quadratic equation

$$a\xi^2 - 2b\xi + c = 0. \tag{3.9}$$

This equation has two distinct real solutions ξ^+ and ξ^-. The characteristic curves are the envelopes to these slopes C^+ for ξ^+ and C^- for ξ^-. Their respective equations are

$$\begin{aligned} I^+ :\ \frac{dy}{d\alpha} &= \xi^+\frac{dx}{d\alpha} \quad \text{along } C^+, \\ \\ I^- :\ \frac{dy}{d\beta} &= \xi^-\frac{dx}{d\beta} \quad \text{along } C^-. \end{aligned} \tag{3.10}$$

Let us to return to system (3.6) and write that the determinant of the first and third equation is nil.

$$\begin{vmatrix} A_1y_\sigma - B_1x_\sigma & A_2y_\sigma - B_2x_\sigma \\ A_1u_\sigma + C_1v_\sigma + E_1x_\sigma & A_2u_\sigma + C_2v_\sigma + E_2x_\sigma \end{vmatrix} = 0. \tag{3.11}$$

After rearrangement, we obtain

$$Tu_\sigma + (a\xi - S)v_\sigma + (K\xi - H)x_\sigma = 0, \tag{3.12}$$

in which,

$$T = [AB],\ S = [BC],\ K = [AE],\ H = [BE]. \tag{3.13}$$

This relation holds on C^+ if we identify ξ with ξ^+ and σ with α, and likewise, C^-, if identify ξ with ξ^- and σ with β.

Thus we arrive at the following four characteristic equations:

$$\begin{aligned}
I^+ \quad & y_\alpha - \xi^+ x_\alpha = 0, \\
I^- \quad & y_\beta - \xi^- x_\beta = 0, \\
II^+ \quad & T u_\alpha + (a\xi^+ - S) v_\alpha + (K\xi^+ - H) x_\alpha = 0, \\
II^- \quad & T u_\beta + (a\xi^- - S) v_\beta + (K\xi^- - H) x_\beta = 0,
\end{aligned} \tag{3.14}$$

It is not overemphasizing to stress again that II^+ is valid only on C^+ whose equation is I^+ and II^- on C^- whose equation is I^-.

Remark 1: Unless the system (3.1) is linear, ξ^+ and ξ^- depend on x, y and u, v. Consequently, the characteristics depend on the individual solutions u, v, and so does the hyperbolic character of the system (3.1).

Example 1: Let us consider the case of a one-dimensional unsteady isentropic flow. The unknown functions are the density ρ and the velocity u, and the independent variables are x, t. The flow equations are:

$$\begin{aligned}
& \rho_t + u\rho_x + \rho u_x = 0, \\
& u_t + u u_x + \frac{c^2}{\rho}\rho_x = 0.
\end{aligned} \tag{3.15}$$

We find that $a = 1$, $b = u$, $c = u^2 - c^2$, $ac - b^2 = -c^2 < 0$. Hence there are two characteristic families:

$$x_\alpha = (u + c)t_\alpha, \qquad x_\beta = (u - c)t_\beta \tag{3.16}$$

The characteristic equation for ρ and u become

$$u_\alpha + \frac{c}{\rho}\rho_\alpha = 0, \qquad u_\beta - \frac{c}{\rho}\rho_\beta = 0. \tag{3.17}$$

Equations (3.16) express the fact that the characteristic curves in the (x,t)-plane represent motions of possible disturbances called sound waves whose velocities

$$\frac{dx}{dt} = u + c, \qquad \frac{dx}{dt} = u - c$$

differ from the particle velocity u by the sound velocity $\pm c$.

For an isentropic flow, $(\gamma - 1)\frac{d\rho}{\rho} = 2\frac{dc}{c}$. This result can be used to eliminate ρ from (3.18) and get

$$u_\alpha + \frac{2}{\gamma - 1}c_\alpha = 0, \qquad u_\beta - \frac{2}{\gamma - 1}c_\beta = 0. \tag{3.18}$$

Equations (3.18) can then be integrated to give

$$u + \frac{2}{\gamma - 1}c = r, \qquad u - \frac{2}{\gamma - 1}c = s, \tag{3.19}$$

where r and s are constant along the characteristics C_+ and C_-, respectively. r and s are known as the Riemann invariants. If one of the Riemann invariants is constant throughout the domain, the solution corresponds to a wave motion in a one direction *only* and is said to be a *simple wave.*

Proposition: A flow in a region adjacent to a region of constant state is always a simple wave.

Example 2: The equations for steady two-dimensional isentropic irrotational inviscid flow are

$$\rho u_x + \rho u_y + u\rho_x + v\rho_y = 0, \tag{3.20}$$

$$uu_x + vu_y + \frac{c^2}{\rho}\rho_x = 0, \tag{3.21}$$

$$uv_x + vv_y + \frac{c^2}{\rho}\rho_y = 0. \tag{3.22}$$

The density ρ can be eliminated from these equations by multiplying the first by $-\frac{c^2}{\rho}$, the second by u and the third by v, and adding them. The resulting equation is

$$(u^2 - c^2)u_x + uvu_y + uvv_x + (v^2 - c^2)v_y = 0. \tag{3.23}$$

To this equation we add the condition of irrotational flow

$$u_y - vx = 0. \tag{3.24}$$

The system of equations (3.20)–(3.22) is quasilinear (in fact, it is a reducible system). The equations of the characteristics directions is

$$(c^2 - u^2)\xi^2 + 2uv\xi + (c^2 - v^2) = 0. \tag{3.25}$$

The characteristics are real if $u^2 + v^2 > c^2$, or if the fluid is supersonic. In this case, the system of characteristic equations can be written as

$$\begin{aligned} I^+ : &\quad y_\alpha = \xi^+ x_\alpha, \\ I^- : &\quad y_\beta = \xi^- x_\beta, \\ II^+ : &\quad u_\alpha = -\xi^+ v_\alpha, \\ II^- : &\quad u_\beta = -\xi^- v_\beta. \end{aligned} \tag{3.26}$$

The characteristic curves determined from I^+ and I^- are usually called Mach lines.

Theorem 2: Consider a smooth curve $\gamma(t)$ where $u(t)$ and $v(t)$ are freely specified, then the boundary-value problem for the system of differential equations

$$\begin{aligned} L_1 &= A_1 u_x + B_1 u_y + C_1 v_x + D_1 v_y + E_1 = 0, \\ L_2 &= A_2 u_x + B_2 u_y + C_2 v_x + D_2 v_y + E_2 = 0, \end{aligned} \tag{3.27}$$

in which $A_1, A_2, \cdots, E_2$ are known functions of x, y, u, v in the region R bounded by the two characteristics passing though point P and the domain of dependence cut by them from the initial curve $\gamma(t)$ has a unique solution.

3. SECOND ORDER QUASILINEAR PARTIAL DIFFERENTIAL EQUATION

We denote by Φ the dependent variable and x, y the independent variables. The general form of a second order partial differential equation is

$$a\Phi_{xx} + 2b\Phi_{xy} + c\Phi_{yy} + d = 0, \tag{3.28}$$

in which a, b, c, d are known functions of x, y, Φ, Φ_x, Φ_y. This problem can be reduced to that of (3.1) by introducing the variables

$$u = \Phi_x, \qquad v = \Phi_y$$

and the fact that for continuously differentiable functions

$$u_y = v_x$$

The equation for the characteristics is

$$a y_\sigma^2 - 2b x_\sigma y_\sigma + c x_\sigma^2 = 0, \tag{3.29}$$

where a, b, c are defined as previously. Calculating the relationship between u_σ and v_σ as in (3.12), we finally arrive at the following characteristic system

$$\begin{aligned} I^+ \quad & y_\alpha - \xi^+ x_\alpha = 0, \\ I^- \quad & y_\beta - \xi^- x_\beta = 0, \\ II^+ \quad & u_\alpha + \xi^- v_\alpha + x_\alpha \frac{d}{a} = 0, \\ II^- \quad & u_\beta + \xi^+ v_\beta + x_\beta \frac{d}{a} = 0. \end{aligned} \tag{3.30}$$

Canonical form of a second order partial differential equation

Note that the two families of characteristic I^+ and I^- can be defined as

$$x = x(\alpha, \beta) \tag{3.31}$$

$$y = y(\alpha, \beta) \tag{3.32}$$

where β is constant along I^+ and α is constant along I^-. If the Jacobian of the transformation (3.31), (3.32) is not zero, we can define

$$\alpha = \alpha(x, y) \tag{3.33}$$

$$\beta = \beta(x, y). \tag{3.34}$$

Using the new variables α and β, equation (3.28) becomes

$$\Phi_{\alpha,\beta} = F(\alpha, \beta, \Phi_\alpha, \Phi_\alpha, \Phi). \tag{3.35}$$

Equation (3.35) in known as the canonical form of (3.28).