UNIVERSITY OF NOTRE DAME
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

Professor H.M. Atassi
113 Hessert Center
Tel: 631-5736
Email:atassi.1@nd.edu

AME-60612
Mathematical Methods II

Linear Operators and Linear Equations

1 Linear Equations

Let E be an n-dimensional space and let $\vec{x} = \{x_1, x_2, \cdots, x_n\} \in E$. We define the inner product

$$(\vec{x}, \vec{y}) = \sum_{i=1}^{i=n} x_i y_i. \tag{1}$$

and the norm

$$||\vec{x}|| = (\vec{x}, \vec{x})^{\frac{1}{2}}. \tag{2}$$

A linear operator $\mathcal{L}$ on E is denoted $\mathcal{L}$: $E \to F \subseteq E$. In linear algebra, the operator can be represented by a matrix $\mathcal{A}$. An $n \times n$ matrix $\mathcal{A}$ can be represented by its column vectors, $\mathcal{A} = \{c_1, c_2, \cdots, c_n\}$ or by its row vectors $\mathcal{A} = \{r_1, r_2, \cdots, r_n\}^t$. The rank of $\mathcal{A}$ is the number of independent column vectors or the number of independent row vectors. If the rank of $\mathcal{A}$ is $< n$, $\mathcal{A}$ is said to be *singular*.

A system of n linear equations can be written as

$$\mathcal{A}\vec{x} = \vec{b}. \tag{3}$$

Or

$$\sum_{i=1}^{i=n} x_i \vec{c_i} = \vec{b}. \tag{4}$$

1.1 $\mathcal{A}$ is of rank n

The column vectors $\{\vec{c_i}\}$ are independent and span E. The associated homogenous equation, $\mathcal{A}\vec{x} = 0$, has no nontrivial solutions. Equation (3) has a unique solution

$$\vec{x} = \mathcal{A}^{-1}\vec{b}. \tag{5}$$

The matrix $\mathcal{A}$, or the operator $\mathcal{L}$, is said to be invertible.

1.2 $\mathcal{A}$ is singular and of rank $n-k<n$

The column vectors span a subspace S_{n-k} of dimension $n-k$. The homogeneous equation associated with (3), $\mathcal{A}\vec{x}=0$, implies that $\vec{r}_i\cdot\vec{x}=0$ for $i=1,2,\cdots,n$. As a result the Null space of $\mathcal{A}$, denoted $\mathcal{N}(\mathcal{A})$, is orthogonal to all $\vec{r}_i$ for $i=1,2,\cdots,n$. Similarly, $\mathcal{N}(\mathcal{A}^t)$ is orthogonal to all $\vec{c}_i$ for $i=1,2,\cdots,n$.

1. If $\vec{b}\notin S_{n-k}$, equation 3 has no solution.
2. If $\vec{b}\in S_{n-k}$, then $\vec{b}$ is orthogonal to $\mathcal{N}(\mathcal{A}^t)$. The solution of equation 3 is

$$\vec{x}=\vec{x}_h+\vec{x}_p \tag{6}$$

where $\vec{x}_p$ is a particular solution of 3 and $\vec{x}_h$ is a solution of the associated homogenous equation given by

$$\vec{x}_h=\sum_{i=1}^{i=k}a_i\vec{e}_i \tag{7}$$

where $\vec{e}_i$ are independent vectors which span $\mathcal{N}(\mathcal{A})$ and a_i are arbitrary constants.

2 Hilbert Space

Let E be an infinite dimensional vector space and let $\vec{x}=\{x_1,x_2,\cdots,x_n,\cdots\}\in E$. For example, consider the linear vector space of all functions $f(t)$ continuous on the closed interval [a, b]. We denote such a vector space by C(a, b).

1. **Normed Space** : E is said to be a normed vector space if a norm $||\vec{x}||$ is defined in E. For example, in C(a, b), we define the norm as the maximum value of the function in the interval (a, b).
2. **Convergence and Complete Space**: A sequence of vectors $\vec{x}_1,\vec{x}_2,\cdots,\vec{x}_n,\cdots$, in E is said to converge to a vector $\vec{x}$ in E if, given $\epsilon>0$, there exists an integer $N(\epsilon)$ such that

$$||\vec{x}-\vec{x}_n||<\epsilon, \tag{8}$$

for all $n>N$. It follows that for a given $\epsilon>0$, there exists an integer $N_1(\epsilon)$ such that

$$||\vec{x}_m-\vec{x}_n||<\epsilon, \tag{9}$$

for all n and m greater than N_1. Such a sequence is known as a Cauchy sequence. The vector $\vec{x}$ is called the *limit* of the sequence, and we write

$$\vec{x}=\lim_{n\to\infty}\vec{x}_n. \tag{10}$$

A vector space E is said to be *complete* if for every Cauchy sequence $\{\vec{x}_n\}$ in E there exists a vector $\vec{x}\in E$ such that $\vec{x}=\lim_{n\to\infty}\vec{x}_n$.

3. **Inner Product**: For finite dimensional spaces we have defined the inner product of two vectors by (1). However, for an infinite dimensional space such a product will tend to infinity and thus is meaningless. Various inner products are usually used depending on the spaces. For example, for two functions u and v in C(0, 1), we use the inner product

$$(u, v) = \int_0^1 u(t)v(t)dt, \tag{11}$$

and the norm $||u|| = (u, u)^{\frac{1}{2}}$, where u and $v \in C(0, 1)$.

Definition: A complete normed linear space with an inner product is called a *Hilbert space.*

3 Sturm-Liouville Theory

Consider the second order differential operator in self-adjoint form

$$\mathcal{L} \equiv \frac{d}{dx}(p(x)\frac{d}{dx}) + q(x), \tag{12}$$

where $p(x) \in C^1(a, b)$ and $q(x) \in C(a, b)$. Consider the homogeneous equation

$$\mathcal{L}u = \lambda r(x)u \tag{13}$$

where $r(x) \in C(a, b)$ and λ is a constant. The constant λ for which a nontrivial solution to (13) exists is called an *eigenvalue* of $\mathcal{L}$, and the solution $u(x) \in C^2(a, b)$ corresponding to λ is called an *eigenfunction.*

Definition: If $(\mathcal{L}u, v) = (u, \mathcal{L}^*v)$ for all u and $v \in C^2(a, b)$, then $\mathcal{L}^*$ is called the *adjoint operator* of $\mathcal{L}$.

Definition: An operator $\mathcal{L}$ is said to be *self-adjoint* if $\mathcal{L}^* = \mathcal{L}$.

Theorem: Every pair of eigenfunctions belonging to distinct eigenvalues of a self-adjoint operator $\mathcal{L} : S \subseteq C^2(a, b) \to C(a, b)$ are orthogonal with respect to the weight function $r(x)$.

Proof: $\mathcal{L}u = \lambda ru$ and $\mathcal{L}v = \mu rv$. Consider the inner products, $(v, \mathcal{L}u) = \lambda(ru, v)$ and $(u, \mathcal{L}v) = \mu(rv, u)$. Since the operator is self-adjoint, the two inner products are equal. Therefore, $\lambda(ru, v) = \mu(rv, u)$. Or $(\lambda - \mu)(ru, v) = 0$. Since $\lambda \neq \mu$, $(ru, v) = 0$.

3.1 Conditions for $\mathcal{L}$ to be Self–Adjoint

If $\mathcal{L}$ is self–adjoint, we have

$$(\mathcal{L}u, v) = (u, \mathcal{L}v) \tag{14}$$

for all functions u and $v \in \mathcal{S}$. Substituting the expression for $\mathcal{L}$ from 13, we get

$$\int_a^b [\frac{d}{dx}(p\frac{du}{dx}) + qu]vdx = \int_a^b u[\frac{d}{dx}(p\frac{dv}{dx}) + qv]dx \tag{15}$$

which, after integration by parts, yields the condition

$$[p(uv' - u'v)]_a^b = 0. \tag{16}$$

where the expression $[X]_a^b = X(b) - X(a)$. This condition shows that the condition for an operator to be self–adjoint depends on the property of the function space $\mathcal{S}$ as well as on the operator through the function $p(x)$. The following shows typical conditions to be satisfied for self–adjointness.

1. $u(a) = 0, u(b) = 0$, for all $u \in \mathcal{S}$.
2. $u'(a) = 0, u'(b) = 0$, for all $u \in \mathcal{S}$.
3. $u'(a) - \sigma_1 u(a) = 0, u'(b) - \sigma_2 u(b) = 0$, for all $u \in \mathcal{S}$.
4. $u(a) = u(b)$, and, $p(a)u'(a) = p(b)u'(b)$.
5. $u(a)$ and $u'(a)$ are finite and $p(a) = 0$, and $u(b)$, and, $u'(b)$ are finite and $p(b) = 0$.

Note that the first three conditions are *homogeneous* boundary conditions which defines the function space $\mathcal{S}$ and are independent of the coefficients of the operator. Conditions (4) are *periodic* and impose conditions on the coefficient $p(x)$ of the operator. Finally, conditions (5) is an example of homogeneous boundary conditions satisfied by the coefficient $p(x)$ of the operator.

These examples clearly shows that self–adjointness is a property of the *boundary value problem* rather than that of the operator.

Theorem: Eigenfunctions corresponding to distinct eigenvalues of a self-adjoint operator $\mathcal{L} : S \subseteq C^2(a,b) \to C(a,b)$ are orthogonal with respect to the weight function $r(x)$ and, for *homogeneous boundary conditions* eigenfunctions corresponding to the same eigenvalue are *not* independent.

Proof: We only give proof to the last statement. If $\mathcal{L}u = \lambda ru$ and $\mathcal{L}v = \lambda rv$, then

$$v\mathcal{L}u - u\mathcal{L}v = [p(vu' - uv')]'.$$

Integrating gives, $p(vu' - uv') = Constant$. For homogeneous conditions the constant is zero, and we have, $vu' - uv' = 0$, or

$$\frac{u'}{u} = \frac{v'}{v},$$

which after integration gives, $u = Kv$, where K is a constant.

Definition: A set of functions $\{\varphi_0, \varphi_1, \cdots, \varphi_n, \cdots, \}$ is a basis for C[a, b] if the set is linearly independent and if every element $f \in C[a, b]$ may be written as a linear combination of the set. Such a set of functions is called a *complete* set of functions.

Theorem: The eigenfunctions $\{\varphi_0, \varphi_1, \cdots, \varphi_n, \cdots, \}$ of a self-adjoint operator $\mathcal{L} : S \subseteq C^2(a, b) \to C(a, b)$ form a *complete* set.

Corollary: If a set of functions is complete in a space then any function in the space can be written as a linear combination of the elements of the complete set. Thus for any square integrable continuous function $f(x)$ we can write

$$f(x) = \sum_{n=0}^{n=\infty} a_n \varphi_n(x), \tag{17}$$

where $\varphi_n(x)$ are the eigenfunctions of the adjoint operator.

Theorem: Any regular self–adjoint boundary–value problem has an infinite sequence of *real* eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 \cdots$ with

$$\lim_{n\to\infty} \lambda_n = \infty.$$

The eigenfunction $\varphi_n(x)$ corresponding to the eigenvalue λ_n has exactly n zeros in the interval $[a, b]$.

Example: Consider the boundary–value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \; y(0) = 0, \; y(\pi) = 0,$$

whose solution is $\lambda_n = (n+1)^2, n = 1, 2, \cdots$ and $\varphi_n = sin(n+1)x$. Then $\varphi_0 = sinx$ has no zeros between 0 and π, while $\varphi_n = sin(n+1)x$ has n zeros between 0 and π.

Theorem: For a regular self–adjoint boundary–value problem, the eigenvalues λ_n are given by the asymptotic formula

$$\sqrt{\lambda_n} = \frac{n\pi}{b-a} + \frac{O(1)}{n} \text{for} n = 1, 2, \cdots.$$

4 Existence and Uniqueness of the Solution of $\mathcal{L}y = f$

Theorem: If the homogeneous equation $\mathcal{L}y = 0$ has a non-trivial solution, the solution of the corresponding non-homogeneous equation is not unique. Conversely. if the solution of the non-homogeneous equation is not unique, there exists a non-trivial solution of the homogeneous equation.

Theorem: The non-homogeneous equation

$$\mathcal{L}y = f \tag{18}$$

has a solution for a given function f if, and only if, f is orthogonal to the null space of the adjoint homogeneous equation

$$\mathcal{L}^* z = 0. \tag{19}$$

That is if

$$(f, z) = 0 \tag{20}$$