Kinematic Waves

Introduction

These are waves which result from the conservation equation

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{I} = 0 \tag{1}$$

where E represents a scalar density field and $\mathbf{I}$, its outer flux. The one-dimensional form of (1) is

$$\frac{\partial E}{\partial t} + \frac{\partial I}{\partial x} = 0. \tag{2}$$

As an example, we consider a gas of density ρ streaming with a velocity $\mathbf{v}$. In this case, the conservation of mass can be expressed by (1), if we take $E = \rho$ and $\mathbf{I} = \rho\mathbf{v}$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0. \tag{3}$$

For a one-dimensional problem, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0. \tag{4}$$

We note that neither (3) nor (4) are sufficient to determine both ρ and v, since we have one equation and more than one variable.

We now confine our attention to the one-dimensional equation (2) and further assume that the flux depends only on the density E., i.e., $I = I(E)$. We can the write

$$\frac{\partial E}{\partial t} + \frac{dI}{dE}\frac{\partial E}{\partial x} = 0. \tag{5}$$

It is customary to take the following notations: $E = u$, $c(u) = \frac{dI}{dE}$, and to write (5) as

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0 \tag{6}$$

Equation (6) is a first–order *quasi-linear* partial differential equation.

General Solution:

The characteristics equations for (6) are given by

$$\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0}. \tag{7}$$

The last equation shows that along a characteristic curve,

$$u = k_2. \tag{8}$$

The first equation of (7) then reduces to

$$\frac{dt}{1} = \frac{dx}{c(k_2)}, \tag{9}$$

which can be integrated immediately to give

$$x - c(k_2)t = k_1. \tag{10}$$

Therefore the general solution is given in an implicit form,

$$u = f[x - c(u)t]. \tag{11}$$

The equation

$$\frac{dx}{dt} = c(k_2) \tag{12}$$

represents the projection of the characteristic on the x-t plane. Let $\mathcal{C}$ be this projection which is here a straight line that intersects the x–axis at ξ and along which u is constant.

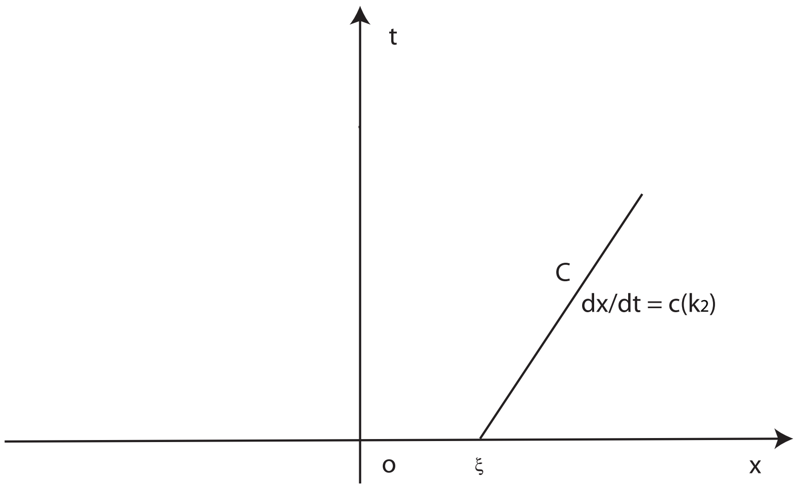

Figure 1: The projection of the characteristics on the t–x plane.

Initial Value Problem:

To find a solution to (6), we need to specify the values of the function u at some time $t = 0$, for example. Let

$$u(x, 0) = f(x), \qquad -\infty < x < +\infty. \tag{13}$$

The equation for $\mathcal{C}$ is given by

$$x = \xi + c(u)t \tag{14}$$

$\mathcal{C}$ intersects the x-axis at ξ, i.e.,

$$\text{at} \quad t = 0, \qquad x = \xi, \qquad u = f(\xi), \tag{15}$$

Since u is constant along $\mathcal{C}$ and depends only on ξ, the slope of $\mathcal{C}$ is $c(u) = c[f(\xi)]$ is a known function of ξ. Thus we have the following parametric representation of the solution:

$$x = \xi + c[(f(\xi)]t \tag{16}$$
$$u = f(\xi) \tag{17}$$

It is easy to verify that (16 and 17) represent a solution to the initial–value problem defined by (6 and 13). First we note that

$$\frac{\partial u}{\partial t} = f'(\xi)\frac{\partial \xi}{\partial t}, \qquad \frac{\partial u}{\partial x} = f'(\xi)\frac{\partial \xi}{\partial x}.$$

To evaluate $\frac{\partial \xi}{\partial t}$ and $\frac{\partial \xi}{\partial x}$, we first differentiate (16) with respect to t

$$0 = \frac{\partial \xi}{\partial t} + \frac{dc(u)}{du}\frac{du}{d\xi}\frac{\partial \xi}{\partial t}t + c[f(\xi)], \tag{18}$$

which gives

$$\frac{\partial \xi}{\partial t} = -\frac{c[f(\xi)]}{1 + \frac{dc(u)}{du}f'(\xi)t}. \tag{19}$$

We then differentiate (16) with respect to x,

$$1 = \frac{\partial \xi}{\partial x} + \frac{dc(u)}{du}\frac{du}{d\xi}\frac{\partial \xi}{\partial x}t, \tag{20}$$

which gives

$$\frac{\partial \xi}{\partial x} = \frac{1}{1 + \frac{dc(u)}{du}f'(\xi)t}. \tag{21}$$

We finally get,

$$\frac{\partial u}{\partial t} = -\frac{c[f(\xi)]f'(\xi)}{1 + \frac{dc(u)}{du}f'(\xi)t}, \qquad \frac{\partial u}{\partial x} = \frac{f'(\xi)}{1 + \frac{dc(u)}{du}f'(\xi)t}, \tag{22}$$

which substituted in (6) gives an identity.

Validity of Solution

We now examine whether the solution (16 and 17) to the initial–value problem (6, 13) is valid for all time $t \geq 0$ or whether there is a limit t_b where the solution will break down if $t > t_b$. The expressions for the derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ given by equations (22) suggest that those derivatives may become singular for $t = -1/[(dc/du)f'(\xi)]$.

On the other hand, let us recall that the theorem for the existence and uniqueness of the Cauchy boundary-value problem states that a characteristic $\mathcal{C}$ must intersect the x–axis only once. So, since (6) is nonlinear, the characteristics $\mathcal{C}$ depend on the initial conditions. If, for example, two characteristics intersect this will violate the conditions stated in the existence and uniqueness theorem. Let us then examine what happens when two characteristics intersect each other. At the intersection both characteristics have the same x and t as shown in Figure 2. We consider two characteristics one intersecting the x–axis at ξ, the other at $\xi + \delta\xi$. The equations for these characteristics are given below.

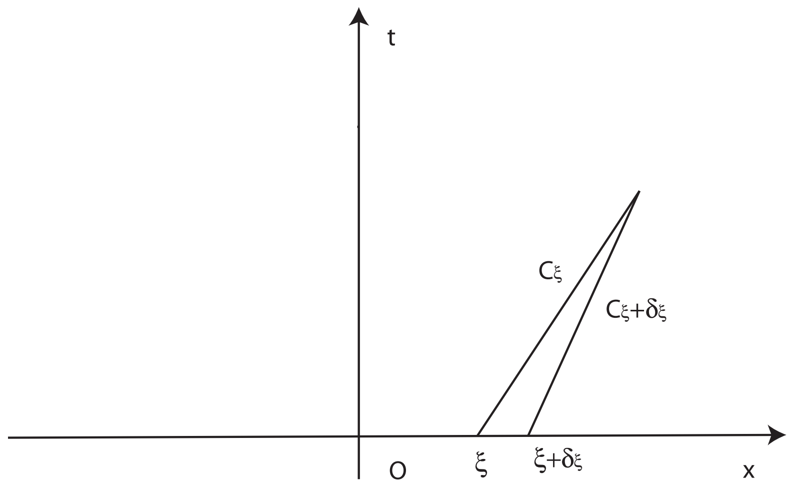

Figure 2: Intersection of two characteristics.

$$
\begin{aligned}
x &= \xi + c[f(\xi)]t, && (23)\\
x &= \xi + \delta\xi + c[f(\xi + \delta\xi)]t = \xi + \delta\xi + c[f(\xi)]t + \frac{dc}{du}f'(\xi)\delta\xi t, && (24)
\end{aligned}
$$

where we have expanded (24) with respect to the small parameter $\delta\xi$. Subtracting (23) from (24), we get

$$
t = -\frac{1}{\frac{dc}{du}f'(\xi)}. \tag{25}
$$

This is exactly the time at which both $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial x}$ become singular. Since t > 0, there is no intersection if $\frac{dc}{du}f'(\xi) > 0$. On the other hand, if

$$\frac{dc}{du}f'(\xi) < 0, \tag{26}$$

the time at first *breaking* will occur at the maximum value of $[\frac{dc}{du}f'(\xi)]$,

$$t_b = \frac{1}{|\frac{dc}{du}f'(\xi)|_{max}}. \tag{27}$$

These results can also be directly derived from the general form of the solution (11). Differentiating u with respect to t, we get

$$\frac{\partial u}{\partial t} = -[c(u) + tdc(u)/du\frac{\partial u}{\partial t}]f'[x - c(u)t], \tag{28}$$

and

$$\frac{\partial u}{\partial t} = -\frac{c(u)f'}{1 + tf'dc(u)/du}. \tag{29}$$

When $t = -\frac{1}{f'dc(u)/du}$, $\frac{\partial u}{\partial t} \to \infty$.

Evolution of the solution in time:

If $dc(u)/du > 0$, then (22) shows that as time increases a *steepening* of the slope of the initial distribution for $f' < 0$ and a *flattening* for $f' > 0$. On the other hand, if $dc(u)/du < 0$, then (22) shows that as time increases a *flattening* of the slope of the initial distribution for $f' > 0$ and a *steepening* for $f' > 0$. As an example, consider the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{30}$$

with the initial condition at $t = 0$, $u = exp(-x^2/2)$. Here, $c = u$ and $dc/du = 1$. The breaking occurs at $t = 1.6487$. Figure 3 shows the evolution of the solution in time. As time increases the slope of the wave steepens to the right and flattens to the left. For $t > 1.6487$, the solution becomes multivalued and hence physically unacceptable.

Traffic Flow:

Let $\rho(x,t)$ be a density per unit length. The cars are moving at speed $v(x,t)$. The car flux, the number of cars crossing a point per unit time, is : $q(x,t) = \rho v$. The

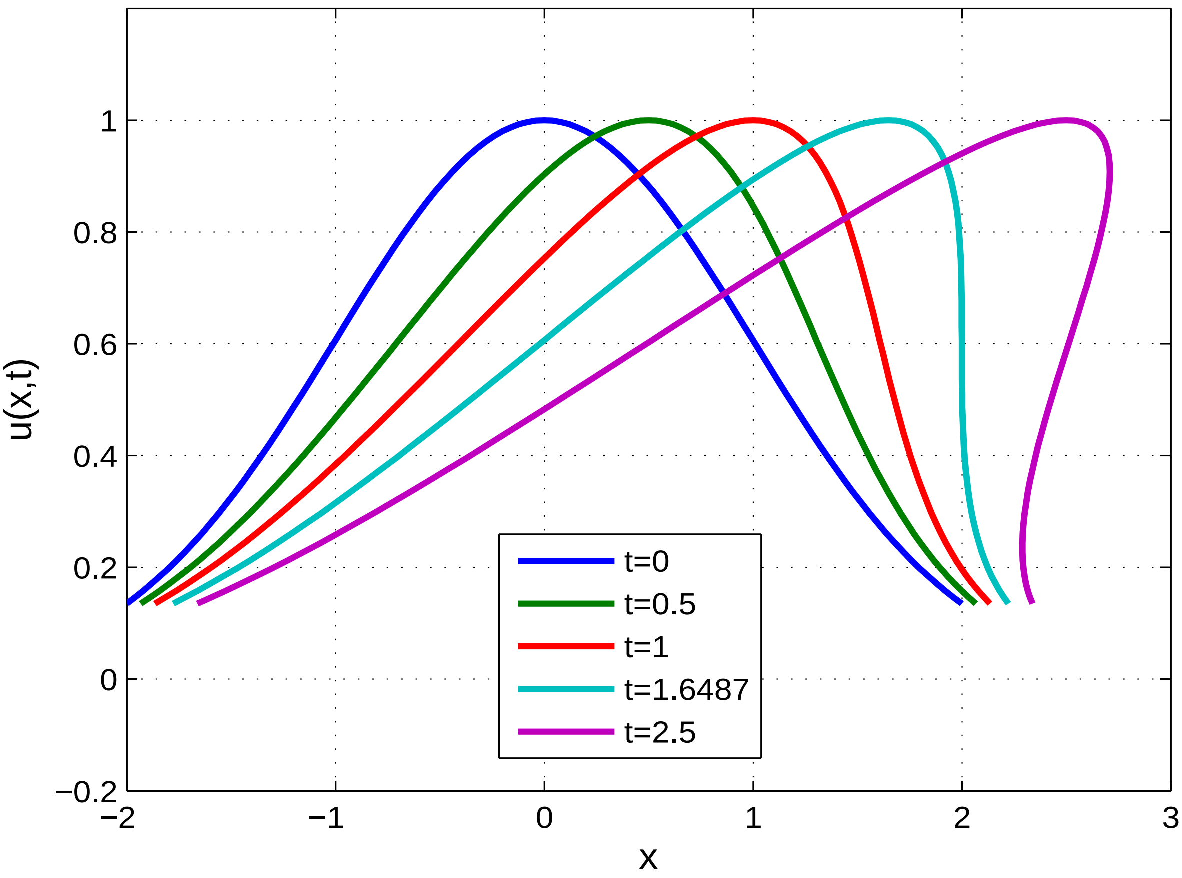

Figure 3: Evolution of the solution in time. Breaking occurs at t=1.6487. Observe how nonlinear effects flatten the wave to the left and steepen it to the right.

conservation relation between ρ and q is summarized as follows

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx = \text{cars in - cars out} \tag{31}$$

$$= q(x_1,t) - q(x_2,t) \tag{32}$$

$$= -\int_{x_1}^{x_2} \frac{\partial q}{\partial x} dx \tag{33}$$

So,

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{34}$$

The wave velocity

$$c(\rho) = \frac{dq}{d\rho} = v + \rho\frac{\partial v}{\partial \rho}. \tag{35}$$

Since $\frac{\partial v}{\partial \rho} < 0$, it follows that $c < v$. Hence for a driver the wave will be moving with the negative velocity $c - v < 0$, implying the wave propagates backward. As an example we consider the traffic through the Lincoln tunnel in New York. Data were gathered, and it was found that

$$\begin{aligned} q &= a\rho(1 - \frac{\rho}{\rho_j}) && (36) \\ a &= 17.2 \text{ mph} && (37) \\ \rho_j &= 250 \text{ vpm} && (38) \\ c(\rho) = \frac{dq}{d\rho} &= a(1 - \frac{2\rho}{\rho_j}) && (39) \\ \frac{dc}{d\rho} &= -\frac{2a}{\rho_j} && (40) \end{aligned}$$

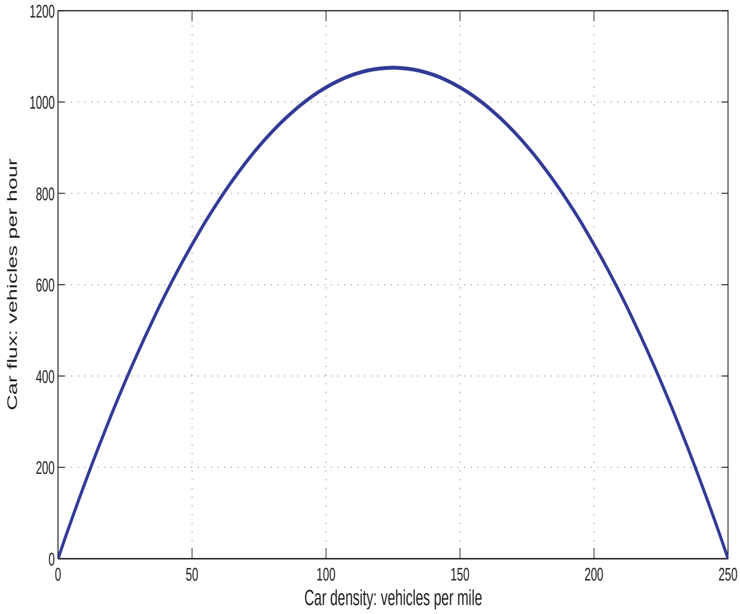

Figure 4: Car flux versus car density in the Lincoln tunnel.

We assume the following initial distribution for the cars.

$$\rho(x, 0) = \bar{\rho} e^{-\alpha x^2} \qquad (41)$$

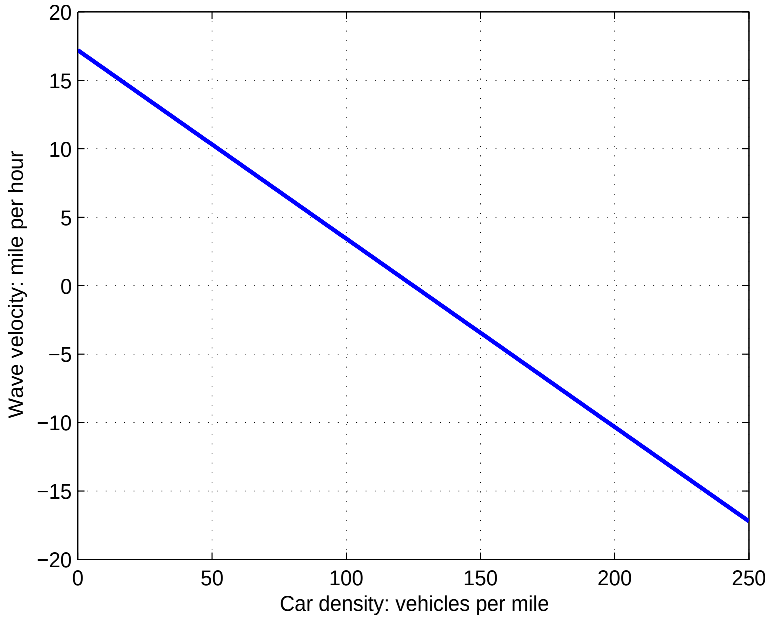

Figure 5: Wave velocity versus car density.

The wave velocity $c(\rho) = dq/d\rho$, hence

$$c[\rho(\xi)] = a(1 - \frac{2\bar{\rho}}{\rho_j} e^{-\alpha\xi^2}) \tag{42}$$

$$\frac{dc}{d\xi} = \frac{4a\alpha\bar{\rho}}{\rho_j} \xi e^{-\alpha\xi^2} \tag{43}$$

$$t_b = [\frac{-1}{c_u'}]_{min} = +\frac{\rho_j}{4a\alpha\bar{\rho}} \frac{1}{|\xi e^{-\alpha\xi^2}|_{max}} \tag{44}$$

Let,

$$z = \xi e^{-\alpha\xi^2} \tag{45}$$

$$\frac{dz}{d\xi} = (1 - 2\alpha\xi^2) e^{-\alpha\xi^2}. \tag{46}$$

The maximum occurs at $\xi = -\sqrt{\frac{1}{2\alpha}}$. Therefore,

$$t_b = \frac{\rho_j}{2a\bar{\rho}} \sqrt{\frac{e}{2\alpha}}. \tag{47}$$

Damped Waves

Consider the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + au = 0, \tag{48}$$

where is a positive constant. The characteristic equations for are (48)

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{-au} \tag{49}$$

Integrating (49), we get

$$\frac{du}{u} = -adt \tag{50}$$

$$u = k_1 e^{-at} \tag{51}$$

$$x = -\frac{k_1}{a}e^{-at} + k_2. \tag{52}$$

Applying the initial condition,

$$u(x,0) = f(x), \qquad u = f(\xi) \tag{53}$$

$$t = 0, \qquad u = k_1 = f(k_2) \tag{54}$$

$$\xi = k_2 - \frac{k_1}{a} \tag{55}$$

$$k_2 = \xi + \frac{f(\xi)}{a} \tag{56}$$

$$x = \xi + \frac{1-e^{-at}}{a}f(\xi) \tag{57}$$

$$u = f(\xi)e^{-at} \tag{58}$$

let us examine the *breaking* of the solution. Two characteristic curves interact

$$x = \xi_1 + \frac{1-e^{-at}}{a}f(\xi_1) \tag{59}$$

$$x = \xi_2 + \frac{1-e^{-at}}{a}f(\xi_2) \tag{60}$$

$$0 = (\xi_2 - \xi_1) + \frac{1-e^{-at}}{a}(f(\xi_2) - f(\xi_1)) \tag{61}$$

$$0 = 1 + \frac{1-e^{-at}}{a}\frac{(f(\xi_2) - f(\xi_1))}{\xi_2 - \xi_1} \tag{62}$$

$$0 = 1 + \frac{1-e^{-at}}{a}f'(\xi). \tag{63}$$

This implies that $f'(\xi) < 0$, and

$$f'(\xi) < -a. \tag{64}$$

The breaking will occur only if the initial curve has enough negative slope. The breaking time is

$$t_b = \{-\frac{\ln(1 + a/f'(\xi))}{a}\}_{min}. \tag{65}$$