UNIVERSITY OF NOTRE DAME
DEPARTMENT OF AEROSPACE AND MECHANICAL ENGINEERING

Professor Atassi *AME-562*

INVERSION OF THE CAUCHY TYPE INTEGRAL

Consider the integral

$$\frac{1}{\pi}\oint_a^b \frac{\varphi(\tau)}{\tau - t} d\tau = f(t) \tag{1}$$

where $b > a$ and $\varphi(t)$ satisfies Holder's condition on the interval $[a, b]$. The inverse problem can be stated as follows:" Given the function $f(t)$ on the interval $[a, b]$, find the density function $\varphi(t)$." The inverse problem is *not* unique. It has the following solutions:

1. Solution bound at a and unbound at b

$$\varphi(t) = -\frac{1}{\pi}\sqrt{\frac{t-a}{b-t}} \oint_a^b \sqrt{\frac{b-\tau}{\tau-a}} \frac{f(\tau)}{\tau-t} d\tau \tag{2}$$

2. Solution bound at b and unbound at a

$$\varphi(t) = -\frac{1}{\pi}\sqrt{\frac{b-t}{t-a}} \oint_a^b \sqrt{\frac{\tau-a}{b-\tau}} \frac{f(\tau)}{\tau-t} d\tau \tag{3}$$

3. Solution unbound at both ends a and b

$$\varphi(t) = -\frac{1}{\pi}\frac{1}{\sqrt{(b-t)(t-a)}} [\oint_a^b \frac{\sqrt{(\tau-a)(b-\tau)}\ f(\tau)}{(\tau-t)} d\tau + C] \tag{4}$$

   where c is an arbitrary constant.

4. Solution bound at both ends a and b

   In general, the inverse solution may not exist. However, if the function $f(t)$ satisfies the condition

$$\oint_a^b \frac{f(\tau)}{\sqrt{(\tau-a)(b-\tau)}} d\tau \tag{5}$$

   then, we have

$$\varphi(t) = -\frac{1}{\pi}\sqrt{(b-t)(t-a)}\ \oint_a^b \frac{f(\tau)}{\sqrt{(\tau-a)(b-\tau)}}\ \frac{d\tau}{(\tau-t)} \tag{6}$$