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Mathematical Methods II

Green's Function

I. Let $r = |\vec{x} - \vec{y}|$, where the position vector $\vec{x} = \{x_1, x_2, x_3\}$ is called the observation point, and $\vec{y} = \{y_1, y_2, y_3\}$ is a constant vector we call the source point. The gradient of $1/r$ is defined in Cartesian coordinates as $\nabla 1/r = \{\partial 1/r/\partial x_1, \partial 1/r/\partial x_2, \partial 1/r/\partial x_3\}$.

It is easy to show that $\nabla^2(\frac{1}{r}) = 0$, except at $r = 0$ since the function is singular at this point. It is interesting to examine the behavior of $\nabla^2(\frac{1}{r})$ at the singular point $\vec{x} = \vec{y}$. Let us consider the integral

$$\int_{\mathcal{V}} \nabla^2(\frac{1}{r}) d\vec{x}, \tag{1}$$

where $\mathcal{V}$ is volume inside the sphere Σ centered at the point $\vec{y}$ and of radius R. Using the divergence theorem, we get

$$\int_{\mathcal{V}} \nabla^2(\frac{1}{r}) d\vec{x} = \int_{\Sigma} \nabla(\frac{1}{r}) \cdot \vec{n} d\sigma_r, \tag{2}$$

where $\vec{n}$ is the unit outward normal to Σ. We note that $\nabla(1/r) = -\vec{n}/r^2$ and that the elementary surface $d\sigma_r = r^2 sin\varphi d\varphi d\theta$. Carrying out the surface integral of the right-hand-side of (2), we get

$$\int_{\mathcal{V}} \nabla^2(\frac{1}{r}) d\vec{x} = -4\pi. \tag{3}$$

The function $g(r) = 1/r$ is known as the free–space Green function for the Laplace equation in a three-dimensional space.

The function

$$\delta(\vec{x} - \vec{y}) = -\frac{1}{4\pi} \nabla^2(\frac{1}{r}) \tag{4}$$

vanishes everywhere except at $\vec{x} = \vec{y}$, and is such that its integral in any sphere is equal to unity. The function $\delta(\vec{x}-\vec{y})$ is known as the Dirac function. Of course, it should be pointed out that we are stretching the definition of functions by calling δ a function. However, δ has some very interesting properties. For example,

$$\int_{\mathcal{V}} \delta(\vec{x} - \vec{y}) f(\vec{x}) d\vec{x} = f(\vec{y}). \tag{5}$$

We also note that δ is an even function, i.e., $\delta(\vec{x} - \vec{y}) = \delta(\vec{y} - \vec{x})$

This result will help obtain a particular solution to the inhomogeneous equation

$$\nabla^2 u = f(\vec{x}). \tag{6}$$

If we express $f(\vec{x})$ using (5), (6) becomes

$$\nabla^2 u = \int_{\mathcal{V}} \delta(\vec{x} - \vec{y}) f(\vec{y}) d\vec{y}, \tag{7}$$

where we have exchanged the variables $\vec{x}$ and $\vec{y}$. Since only δ in the right-hand-side of (7) depends on $\vec{x}$, we deduce immediately,

$$u = \frac{-1}{4\pi} \int_{\mathcal{V}} \frac{f(\vec{y})}{r} d\vec{y}. \tag{8}$$

The Green's function $g(r) = 1/r$ can be thought of as the inverse of the Laplace operator.