## Acoustic Waves in a Circular Duct

Consider a circular duct of radius $a$. We take a cylindrical coordinate system $\{x, r, \theta\}$, where the $x$ axis is along the duct axis. The acoustic pressure is governed by the wave equation

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial^{2} p}{\partial t^{2}}-\nabla^{2} p=0 \tag{1}
\end{equation*}
$$

The pressure must satisfy an initial condition at $x=x_{0}$ and a wall boundary condition at $r=a$. We use the method of separation of variables and assume

$$
\begin{equation*}
p(x, r, \theta, t)=X(x) R(r) \Theta(\theta) T(t) \tag{2}
\end{equation*}
$$

Substituting (2) into (1) and dividing by $X(x) R(r) \Theta(\theta) T(t)$, gives

$$
\begin{equation*}
\frac{X^{\prime \prime}}{X}+\frac{R^{\prime \prime}+R^{\prime} / r}{R}+\frac{\Theta^{\prime \prime}}{r^{2} \Theta}-\frac{1}{c^{2}} \frac{T^{\prime \prime}}{T}=0 \tag{3}
\end{equation*}
$$

If we take

$$
\begin{align*}
& \frac{\Theta^{\prime \prime}}{\Theta}=-m^{2}  \tag{4}\\
& \frac{X^{\prime \prime}}{X}=-k^{2}  \tag{5}\\
& \frac{T^{\prime \prime}}{T}=-\omega^{2} \tag{6}
\end{align*}
$$

where $m$ is an integer. This implies a solution of the form

$$
\begin{equation*}
p_{m k \omega}=R_{m}(r) e^{i(k x+m \theta-\omega t)} \tag{7}
\end{equation*}
$$

The function $R_{m}$ satisfies the equation

$$
\begin{equation*}
r^{2} R_{m}^{\prime \prime}+r R_{m}^{\prime}+\left(\mu^{2} r^{2}-m^{2}\right) R_{m}=0 \tag{8}
\end{equation*}
$$

where we have introduced the eigenvalue $\mu^{2}=\omega^{2} / c^{2}-k^{2}$. For a rigid duct, this equation must satisfy an impermeability condition

$$
\begin{equation*}
\left(\frac{d R_{m}}{d r}\right)_{r=a}=0 \tag{9}
\end{equation*}
$$

Introducing the non-dimensional variable $\tilde{r}=\mu r$, equation (9)becomes

$$
\begin{equation*}
\tilde{r}^{2} \frac{d^{2} R_{m}}{d \tilde{r}^{2}}+\tilde{r} \frac{d R_{m}}{d r}+\left(\tilde{r}^{2}-m^{2}\right) R_{m}=0 \tag{10}
\end{equation*}
$$

We recognize the Bessel equation and since the pressure is finite along the axis $R_{m}=J_{m}(\tilde{r})$. The wall condition (9) implies

$$
\begin{equation*}
J_{m}^{\prime}(\mu a)=0 \tag{11}
\end{equation*}
$$

The boundary-value problem $(10,11)$ is a Sturm-Liouville problem whose solutions form a complete set. The derivative of the Bessel function has an infinite number of zeros which we denote as $\left\{\alpha_{m n}\right\}$,

$$
\begin{equation*}
J_{m}^{\prime}\left(\alpha_{m n}\right)=0, m=0,1, \cdots \tag{12}
\end{equation*}
$$

Hence, the eigenvalues are

$$
\begin{equation*}
\mu_{m n}=\frac{\alpha_{m n}}{a} . \tag{13}
\end{equation*}
$$

This defines the axial wave number as

$$
\begin{equation*}
k_{m n}=\sqrt{\left(\frac{\omega}{c}\right)^{2}-\mu_{m n}^{2}} . \tag{14}
\end{equation*}
$$

The eigenfunction

$$
\begin{equation*}
p_{m n}=J_{m}\left(\frac{\alpha_{m n} r}{a}\right) e^{i\left(k_{m n} x+m \theta-\omega t\right)} \tag{15}
\end{equation*}
$$

is called the $\{m n\}$ mode. For every frequency $\omega$, the solution is then

$$
\begin{equation*}
p_{\omega}=\sum_{m=-\infty}^{m=+\infty} \sum_{n=0}^{n=+\infty} c_{m n} p_{m n} \tag{16}
\end{equation*}
$$

The expression for the coefficients $c_{m n}$ is determined using the initial condition

$$
\begin{equation*}
p_{\omega}(0, r, \theta, t)=f_{\omega}(r, \theta) e^{-i \omega t} \tag{17}
\end{equation*}
$$

and the orthogonality of the Bessel functions,

$$
\begin{equation*}
c_{m n}=\frac{1}{\pi a^{2}} \frac{\alpha_{m n}^{2}}{\left(\alpha_{m n}^{2}-m^{2}\right) J_{m}^{2}\left(\alpha_{m n}\right)} \int_{0}^{2 \pi} \int_{0}^{a} f_{\omega}(r, \theta) J_{m}\left(\alpha_{m n} \frac{r}{a}\right) e^{-i m \theta} r d r d \theta \tag{18}
\end{equation*}
$$

where we have used (see Hildebrand, p. 229)

$$
\begin{equation*}
\int_{0}^{a} r J_{m}^{2}\left(\alpha_{m n} \frac{r}{a}\right) d r=\frac{a^{2}\left(\alpha_{m n}^{2}-m^{2}\right)}{2 \alpha_{m n}^{2}} J_{m}^{2}\left(\alpha_{m n}\right) . \tag{19}
\end{equation*}
$$

Note the condition for propagation of an acoustic mode is that the wave number $k_{m n}$ must be real. Otherwise the wave will decay exponentially and is known as an evanescent wave. Therefore an $\{m n\}$ mode propagates if

$$
\begin{equation*}
\frac{\omega a}{c}>\alpha_{m n} . \tag{20}
\end{equation*}
$$

At low frequencies, only the fundamental mode

$$
\begin{equation*}
p_{00}=e^{i[(\omega / c) x-\omega t]} \tag{21}
\end{equation*}
$$

propagates. As $\omega$ increases an additional mode propagates. The frequency at which a mode $\{m n\}$ begins to propagate is known as the cutoff frequency of the mode. As the frequency increases (decreases) and is equal to the cutoff frequency of a mode $\{m n\}$, the mode $\{m n\}$ is said to cut on (cut off).

As an example, consider a duct of radius $a=0.5 \mathrm{~m}, \mathrm{c}=340 \mathrm{~m} / \mathrm{s}$, and the sound frequency is 3000 rpm . $a \omega / c=0.462$. From the tables of zeros of Bessel functions, the lowest zero is $\alpha_{11}=1.8412$. hence only the fundamental mode will propagate.

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## Bessel Function Zeros

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The first $k$ roots $x_{1}, \ldots, x_{k}$ of the Bessel function $J_{n}(x)$ are given in the following table. They can be found in Mathematica using the command BesselJZeros [ $n, k]$ in the Mathematica add-on package NumericalMath BesselZeros` (which can be loaded with the command <<NumericalMath ) .

| zero | $J_{0}(x)$ | $J_{1}(x)$ | $J_{2}(x)$ | $J_{3}(x)$ | $J_{4}(x)$ | $J_{5}(x)$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 1 | 2.4048 | 3.8317 | 5.1356 | 6.3802 | 7.5883 | 8.7715 |
| 2 | 5.5201 | 7.0156 | 8.4172 | 9.7610 | 11.0647 | 12.3386 |
| 3 | 8.6537 | 10.1735 | 11.6198 | 13.0152 | 14.3725 | 15.7002 |
| 4 | 11.7915 | 13.3237 | 14.7960 | 16.2235 | 17.6160 | 18.9801 |
| 5 | 14.9309 | 16.4706 | 17.9598 | 19.4094 | 20.8269 | 22.2178 |

The first $k$ roots $x_{1}, \ldots, x_{k}$ of the derivative of the Bessel function $J_{x}^{\prime}(x)$ can be found in Mathematica using the command BesseluPrimezeros $[n, k]$ in the Mathematica add-on package NumericalMath 'Besselzeros (which can be loaded with the command <<NumericalMath '). The first few such roots are given in the following table.

| zero | $J_{0}{ }^{\prime}(x)$ | $J_{1}^{\prime}(x)$ | $J_{2}^{\prime}(x)$ | $J_{3}^{\prime}(x)$ | $J_{4}{ }^{\prime}(x)$ | $J_{5}^{\prime}(x)$ |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 3.8317 | 1.8412 | 3.0542 | 4.2012 | 5.3175 | 6.4156 |
| 2 | 7.0156 | 5.3314 | 6.7061 | 8.0152 | 9.2824 | 10.5199 |
| 3 | 10.1735 | 8.5363 | 9.9695 | 11.3459 | 12.6819 | 13.9872 |
| 4 | 13.3237 | 11.7060 | 13.1704 | 14.5858 | 15.9641 | 17.3128 |
| 5 | 16.4706 | 14.8636 | 16.3475 | 17.7887 | 19.1960 | 20.5755 |

SEE ALSO: Bessel Function, Bessel Function of the First Kind. [Pages Linking Here]

## CITE THIS AS:

Eric W. Weisstein. "Bessel Function Zeros." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/BesselFunctionZeros.html

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