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AME-60612
Mathematical Methods II

Homework 1

I. The position vector of a point P is usually defined by its Cartesian coordinates $\{x, y, z\}$ with respect to an orthogonal frame of reference centered at the point O. The x and y axes are taken in a horizontal plane and the z axis is vertical. Consider the spherical coordinates (r, φ, θ) defined by

$$x = r sin\varphi cos\theta, \tag{1}$$
$$y = r sin\varphi sin\theta, \tag{2}$$
$$z = r cos\varphi, \tag{3}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, φ is the angle of the vector $\overrightarrow{OP}$ with the z axis and θ is the angle of the horizontal projection of $\overrightarrow{OP}$ with the x axis. Sketch the coordinate system and the unit vectors associated with the variables (r, φ, θ). Write the expressions for the unit vectors associated with the spherical coordinates in terms of the Cartesian coordinates unit vectors. Give the expressions for the gradient, divergence and curl in spherical coordinates.

II. Consider the vector field $\vec{V} = \{v_x, v_y, v_z\}$:

$$v_x = \frac{x}{(x^2 + y^2 + z^2)^{n/2}}, \quad v_y = \frac{y}{(x^2 + y^2 + z^2)^{n/2}}, \quad v_z = \frac{z}{(x^2 + y^2 + z^2)^{n/2}}, \tag{4}$$

where n is an integer.

1. Show that (4) can also be represented as

$$\vec{V} = \frac{\vec{e_r}}{r^{(n-1)}}, \tag{5}$$

where $\vec{e}_r$ is the unit vector in the radial direction.

2. Calculate $\nabla \cdot \vec{V}$. For what value of n, $\nabla \cdot \vec{V} = 0$.
3. Calculate $\nabla \times \vec{V}$.
4. Is $\vec{V}$ irrotational? If so, find the potential function ϕ so that $\vec{V} = \nabla\phi$. For what value of n, $\nabla^2\phi = 0$. Compare with the result of item 2.
5. Calculate the flux of $\vec{V}$ across the sphere Σ centered at the origin O and of radius R.
6. If $\mathcal{V}$ is the volume inside Σ, calculate the integral

$$\int_{\mathcal{V}} (\nabla \cdot \vec{V}) d\mathcal{V}. \tag{6}$$

7. Use the divergence theorem to verify the results of 2 and 3.

III. Problems 44, 45, 46, 47, 51, 52; page 328.

IV. Problems 56, 57, 60, 61; page 329.