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Advanced Aerodynamics

Flow Induced by Vorticity

1 Free-Space Green Function for Laplace Equation

The Green function, $g(\vec{x}, \vec{y})$, is a solution of the equation

$$\nabla^2 g(\vec{x}, \vec{y}) = -4\pi\delta(\vec{x} - \vec{y}), \tag{1}$$

where $\delta(\vec{x})$ is the Dirac function. Let $r = |\vec{x} - \vec{y}|$. Then, since g depends only on r, $\nabla^2 \equiv \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}$. Equation (1) reduces to

$$\frac{d^2 g}{dr^2} + \frac{2}{r}\frac{dg}{dr} = 0, \quad \text{for} \quad r \neq 0. \tag{2}$$

A solution to (2) is

$$g(r) = \frac{K}{r}, \tag{3}$$

where K is a constant. We use the divergence theorem to determine K,

$$\int_{\mathcal{V}} \nabla^2 (\frac{K}{r}) d\mathcal{V} = \int_{\Sigma} \nabla(\frac{K}{r}).\vec{n} d\Sigma = -K \int_{\Sigma} \frac{(\vec{x} - \vec{y}).\vec{n}}{|\vec{x} - \vec{y}|^3} d\Sigma = -4K\pi, \tag{4}$$

where Σ is a sphere of radius R centered on $\vec{y}$, and $\vec{n}$ is the outward unit vector normal to Σ. The volume integral of the right hand side of (1) is -4π. Therefore, $K = 1$, and we have

$$\nabla^2 (\frac{1}{r}) = -4\pi\delta(\vec{x} - \vec{y}) \tag{5}$$

2 Poisson's Equation

Consider the inhomogeneous equation,

$$\nabla^2 \vec{V} = -4\pi \vec{f}. \tag{6}$$

We note that

$$\vec{f}(\vec{x}) = \int_{\mathcal{V}} \vec{f}(\vec{y})\delta(\vec{y} - \vec{x})d\vec{y} \tag{7}$$

and using (5) for every component of $\vec{f}$, we get

$$V(\vec{x}) = \int_{\mathcal{V}} \frac{\vec{f}(\vec{y})}{|\vec{x} - \vec{y}|} d\vec{y}. \tag{8}$$

Taking the curl of (8)

$$\nabla \times V(\vec{x}) = \int_{\mathcal{V}} \nabla(\frac{1}{r}) \times \vec{f}(\vec{y})d\vec{y}. \tag{9}$$

3 Velocity in Terms of the Vorticity

Consider the solution to Poisson's equation

$$\nabla^2 \vec{A} = -\vec{V}. \tag{10}$$

Taking the curl of both sides of (10), we get

$$\nabla^2(\nabla \times \vec{A}) = -\vec{\zeta}, \tag{11}$$

where $\vec{\zeta} = \nabla \times \vec{V}$. A solution to (11) is

$$(\nabla \times \vec{A}) = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\vec{\zeta}(\vec{y})}{|\vec{x} - \vec{y}|} d\vec{y}. \tag{12}$$

Taking the curl of both sides of (12), we get

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{4\pi} \int_{\mathcal{V}} \nabla_{\vec{x}}(\frac{1}{|\vec{x} - \vec{y}|}) \times \vec{\zeta} d\vec{y}. \tag{13}$$

We recall the mathematical identity,

$$\nabla \times (\nabla \times \vec{A}) \equiv \nabla(\nabla . \vec{A}) - \nabla^2 \vec{A}. \tag{14}$$

If we further assume that $\vec{A}$ is solenoidal, i.e., $\nabla \cdot \vec{A} = 0$, then $\vec{V}$ is also solenoidal, i.e., $\nabla \cdot \vec{V} = 0$. In this case (14) reduces to

$$\nabla \times (\nabla \times \vec{A}) \equiv -\nabla^2 \vec{A} = \vec{V}. \tag{15}$$

Substituting this result into (13), we get

$$\vec{V} = \frac{1}{4\pi}\int_{\mathcal{V}} \nabla_{\vec{x}}(\frac{1}{|\vec{x}-\vec{y}|}) \times \vec{\zeta} d\vec{y}. \tag{16}$$

or

$$\vec{V} = \frac{1}{4\pi}\int_{\mathcal{V}} \frac{\vec{\zeta} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} d\vec{y}. \tag{17}$$

This formula for the induced velocity corresponds exactly to the formula of Biot and Savart for the magnetic field induced by a current. The elementary velocity induced by the vorticity in the element of volume $d\vec{y}$ is

$$d\vec{V} = \frac{1}{4\pi}\frac{(\vec{\zeta} d\vec{y}) \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}. \tag{18}$$

4 Vorticity Concentrated in a Vortex Filament

We now consider a vortex filament $\mathcal{C}$. Let σ be the infinitesimal cross-section of the filament orthogonal to the vorticity $\vec{\zeta}$. Since $\nabla \cdot \vec{\zeta} = 0$, $|\vec{\zeta}|\sigma$ = constant along the filament. Moreover, Stokes theorem states that the circulation, Γ, around a circuit surrounding the filament is equal to the flux of the vorticity, i.e.,

$$\Gamma = \zeta\sigma \tag{19}$$

Let $\vec{ds}$ be the elemental arc in the $\vec{\zeta}$ direction, then (18) becomes

$$d\vec{V} = \frac{\sigma}{4\pi}\frac{\vec{ds} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}. \tag{20}$$

and the total induced velocity

$$\vec{V} = \frac{\sigma}{4\pi}\int_{\mathcal{C}} \frac{\vec{ds} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}. \tag{21}$$

If $\vec{\tau} = \vec{\zeta}/|\vec{\zeta}|$, then $\vec{ds} = \vec{\tau} ds$, and we have

$$\vec{V} = \frac{\sigma}{4\pi}\int_{\mathcal{C}} \frac{\vec{\tau} \times (\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} ds. \tag{22}$$

or

$$\vec{V} = \frac{\sigma}{4\pi}\int_{\mathcal{C}} \vec{\tau} \times \nabla_{\vec{y}}(\frac{1}{|\vec{x}-\vec{y}|}) ds. \tag{23}$$