Permanence of Vorticity

The acceleration $\mathbf{a}$ of a fluid particle is given in terms of the material derivative

$$\mathbf{a} = \frac{D}{Dt}\mathbf{V}. \tag{1}$$

Note that

$$\frac{D}{Dt}\mathbf{V} \equiv \frac{\partial \mathbf{V}}{\partial t} + \zeta \times \mathbf{V} + \frac{1}{2}\nabla \mathbf{V^2}. \tag{2}$$

Taking the curl of both sides of (1), noting that

$\nabla \times (\zeta \times \mathbf{V}) \equiv (\mathbf{V}\cdot\nabla)\zeta - (\zeta\cdot\nabla)\mathbf{V} + \zeta(\nabla\cdot\mathbf{V}) - \mathbf{V}(\nabla\cdot\zeta)$, $\nabla\cdot\zeta = 0$, and $\nabla\cdot\mathbf{V} = -(D\rho)/(\rho Dt)$,

we get

$$\frac{D}{Dt}\frac{\zeta}{\rho} = \frac{\zeta}{\rho}\cdot\nabla\mathbf{V} + \frac{1}{\rho}\nabla\times\mathbf{a}. \tag{3}$$

For conservative forces and barotropic fluids, $\nabla \times \mathbf{a} = 0$, and we get

$$\frac{D}{Dt}\frac{\zeta}{\rho} = (\frac{\zeta}{\rho}\cdot\nabla)\mathbf{V} \tag{4}$$

To integrate this equation, we first write (4) in tensor notation

$$\frac{D}{Dt}\frac{\zeta_i}{\rho} = \frac{\zeta_j}{\rho}\cdot\frac{\partial V_i}{\partial x_j} \tag{5}$$

We introduce the Lagrangian coordinates $\mathbf{x^{(0)}} = \{x_i^{(0)}\}$ and note that $\partial/\partial x_j = (\partial x_k^{(0)}/\partial x_j)\partial/\partial x_k^{(0)}$. Equation (5) can then be written as

$$\frac{D}{Dt}\frac{\zeta_i}{\rho} = \frac{\zeta_j}{\rho}\frac{\partial x_k^{(0)}}{\partial x_j}\frac{\partial V_i}{\partial x_k^{(0)}}. \tag{6}$$

We also note that

$$\frac{\partial x_k^{(0)}}{\partial x_j}\frac{\partial x_i}{\partial x_k^{(0)}} = \delta_{ij}, \tag{7}$$

and therefore,

$$\frac{D}{Dt}\{\frac{\partial x_k^{(0)}}{\partial x_j}\frac{\partial x_i}{\partial x_k^{(0)}}\} = 0. \tag{8}$$

Equation (8) can be written as

$$\frac{\partial x_i}{\partial x_k^{(0)}}\frac{D}{Dt}(\frac{\partial x_k^{(0)}}{\partial x_j}) + \frac{\partial x_k^{(0)}}{\partial x_j}\frac{\partial V_i}{\partial x_k^{(0)}} = 0, \tag{9}$$

since $V_i = (Dx_i)/(Dt)$.
Substituting (9) into (6), we get

$$\frac{D}{Dt}\frac{\zeta_i}{\rho} + \frac{\zeta_j}{\rho}\frac{D}{Dt}(\frac{\partial x_k^{(0)}}{\partial x_j})\frac{\partial x_i}{\partial x_k^{(0)}} = 0. \tag{10}$$

Multiplying by $\partial x_\ell^{(0)}/\partial x_i$ and using (7), we get

$$\frac{\partial x_\ell^{(0)}}{\partial x_i}\frac{D}{Dt}\frac{\zeta_i}{\rho} + \frac{\zeta_j}{\rho}\frac{D}{Dt}(\frac{\partial x_k^{(0)}}{\partial x_j})\delta_{k\ell} = 0. \tag{11}$$

Or

$$\frac{D}{Dt}(\frac{\zeta_i}{\rho}\frac{\partial x_\ell^{(0)}}{\partial x_i}) = 0. \tag{12}$$

The Lagrangian coordinates were defined at some location upstream $\mathbf{x}^{(0)}$. At this location the tensor $\partial x_\ell^{(0)}/\partial x_i = \delta_{i\ell}$, hence

$$\frac{\zeta_i}{\rho}\frac{\partial x_\ell^{(0)}}{\partial x_i} = (\frac{\zeta_\ell}{\rho})_0. \tag{13}$$

Multiplying by $\partial x_j/\partial x_\ell^{(0)}$ and using (7), we get

$$\frac{\zeta_i}{\rho}\delta_{ij} = (\frac{\zeta_\ell}{\rho})_0\frac{\partial x_j}{\partial x_\ell^{(0)}}. \tag{14}$$

Or

$$\frac{\zeta_j}{\rho} = (\frac{\zeta_\ell}{\rho})_0\frac{\partial x_j}{\partial x_\ell^{(0)}}. \tag{15}$$

Equation (15) gives the vorticity at $\mathbf{x}$ in terms of the vorticity at the previous location $\mathbf{x}^{(0)}$. Therefore, if the vorticity associated with a fluid particle is zero at some initial location $\mathbf{x_0}$, it will be zero at all subsequent locations. On the other hand, if the vorticity is not zero at the initial location $\mathbf{x_0}$, it will never be zero. This result is known as the *permanence of vorticity.*