THE BIOT-SAVART LAW

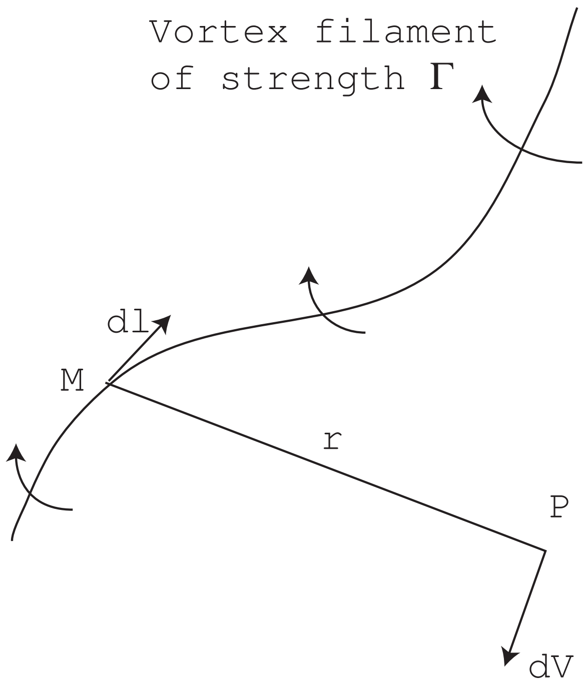

Figure 1: Vortex filament and illustration of the Biot-Savart law.

Consider a vortex filament with a circulation Γ as shown in Figure 1. An elemental segment $d\vec{l}$ centered at the point M of the vortex filament induces an elemental velocity

$$d\vec{V} = \frac{\Gamma}{4\pi} \frac{d\vec{l} \times \vec{r}}{|r^3|}, \tag{1}$$

where $\vec{r} = \overrightarrow{MP}$, and $r = |\vec{r}|$.

We now apply the Biot-Savart law (1) to a straight vortex filament of infinite length as sketched in Figure 2. The velocity $d\vec{V}$ induced at point P by any elemental segment of the vortex filament $d\vec{l}$ is given by (1). Because the filament is a straight line, $d\vec{V}$ is perpendicular to the plane defined by the filament and the point P. The velocity induced at point P by the entire vortex filament is

$$\vec{V} = \frac{\Gamma}{4\pi} \int_{-\infty}^{+\infty} \frac{d\vec{l} \times \vec{r}}{|r^3|}.$$

The direction of the induced velocity can be obtained from the right-hand screw rule. Its magnitude, $V = |\vec{V}|$, can be calculated as follows. From the geometry shown in Figure 2

$$
\begin{aligned}
r &= \frac{h}{sin\theta} \\
l &= -\frac{h}{tan\theta} \\
dl &= \frac{h}{sin^2\theta}d\theta
\end{aligned}
$$

Substituting in equation(2), we have

$$V = \frac{\Gamma}{4\pi}\int_{-\infty}^{+\infty}\frac{sin\theta}{r^2}dl = \frac{\Gamma}{4\pi h}\int_0^{\pi} sin\theta d\theta$$

Or

$$V = \frac{\Gamma}{2\pi h}$$

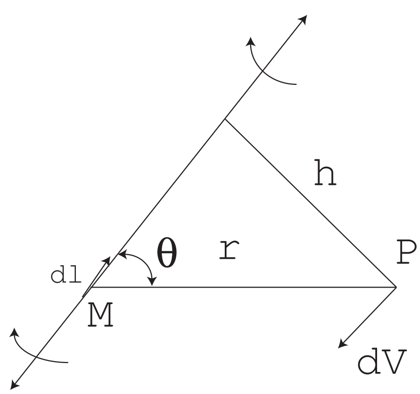

Figure 2: Velocity induced at point P by an infinite straight vortex filament.

Consider the semi-infinite vortex filament shown in Figure 3. The filament extends from O to ∞.

$$V = \frac{\Gamma}{4\pi}\int_{0}^{+\infty}\frac{sin\theta}{r^2}dl = \frac{\Gamma}{4\pi h}\int_{\frac{\pi}{2}}^{\pi} sin\theta d\theta$$

Or

$$V = \frac{\Gamma}{4\pi h}$$

The velocity induced at P by the semi-infinite vortex filament is half that induced by an infinite vortex filament.

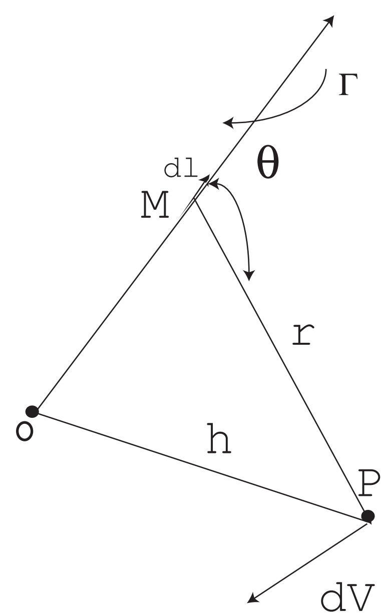

Figure 3: Velocity induced at point P by a semi-infinite straight vortex filament.