Rate of Change of Circulation

Kelvin Theorem: In an inviscid barotropic fluid, the circulation is constant along a circuit moving with the fluid.
The circulation Γ along a circuit $\mathcal{C}$ is defined by the integral

$$\Gamma = \int_{\mathcal{C}} \mathbf{V} \cdot \mathbf{ds}. \tag{1}$$

Taking the material derivative of (1),

$$\begin{aligned} \frac{D}{Dt}\Gamma &= \int_{\mathcal{C}} \frac{D}{Dt}(\mathbf{V} \cdot \mathbf{ds}) & (2) \\ &= \int_{\mathcal{C}} \mathbf{a} \cdot \mathbf{ds} + \int_{\mathcal{C}} \mathbf{V} \cdot \frac{D}{Dt}\mathbf{ds}, & (3) \end{aligned}$$

where $\mathbf{a}$ represents the acceleration of a fluid particle given in terms of the material derivative

$$\mathbf{a} = \frac{D}{Dt}\mathbf{V} = \frac{\partial \mathbf{V}}{\partial t} + \zeta \times \mathbf{V} + \frac{1}{2}\mathbf{V}^2. \tag{4}$$

Note that

$$\frac{D}{Dt}\mathbf{ds} = d\frac{D\mathbf{s}}{Dt} = d\mathbf{V}.$$

Hence the last integral in (3) is a total differential, whose integral along a closed circuit vanishes,

$$\int_{\mathcal{C}} \mathbf{V} \cdot \mathbf{dV} = 0.$$

Thus

$$\frac{D}{Dt}\Gamma = \int_{\mathcal{C}} \mathbf{a} \cdot \mathbf{ds}. \tag{5}$$

If Σ denotes a surface having $\mathcal{C}$ for boundary, then using Stokes theorem gives

$$\frac{D}{Dt}\Gamma = \int_{\Sigma} (\mathbf{B} \cdot \mathbf{n}) d\sigma, \tag{6}$$

where we have put $\mathbf{B} = \nabla \times \mathbf{a}$. From Crocco's equation,

$$\frac{\partial \mathbf{V}}{\partial t} + \zeta \times \mathbf{V} = -\nabla h_0 + T\nabla S. \tag{7}$$

Substituting the expression of $\frac{\partial \mathbf{V}}{\partial t} + \zeta \times \mathbf{V}$ given in (7) into (4) and taking the curl of (4), gives

$$\mathbf{B} = \frac{\partial \zeta}{\partial t} + \nabla \times (\zeta \times \mathbf{V}) = \nabla T \times \nabla S. \tag{8}$$

Therefore,

$$\frac{D}{Dt}\Gamma = \int_{\mathcal{C}} [(\nabla T \times \nabla S) \cdot \mathbf{n})] d\sigma, \tag{9}$$

Euler's equations give

$$\mathbf{a} = \mathbf{f} - \frac{1}{\rho}\nabla p \tag{10}$$

whose curl gives

$$\mathbf{B} = \nabla \times \mathbf{f} - \nabla(\frac{1}{\rho}) \times \nabla p. \tag{11}$$

If $\mathbf{f}$ is conservative, $\mathbf{f} = -\nabla\Omega$ and $\nabla \times \mathbf{f} = 0$. If in addition, $p = f(\rho)$, i.e., the fluid is barotropic, then $\nabla(\frac{1}{\rho}) \times \nabla p = 0$. Hence $\mathbf{B} = 0$, and

$$\frac{D}{Dt}\Gamma = 0. \tag{12}$$