Unsteady Two-Dimensional Thin Airfoil Theory

1 General Formulation

Consider a thin airfoil of infinite span and chord length c. The airfoil may have a small motion about its mean position. Let the x axis be aligned with the airfoil mean position and centered at its midchord O. The flow upstream of the airfoil may have small nonuniformities but its mean velocity $\mathbf{U}$ is uniform and in the x-direction. The velocity field can be written as

$$\mathbf{V}(x,y,t) = \mathbf{U} + \mathbf{u}_\infty(x,y,t) + \mathbf{u}(x,y,t), \tag{1}$$

where $\mathbf{u}_\infty(x,y,t)$ represents the velocity of the imposed upstream nonuniformities, and $\mathbf{u}(x,y,t)$ stands for the perturbation velocity generated by the presence of the airfoil. The assumption of small airfoil motion and small upstream nonuniformities imply that $|\mathbf{u}_\infty(x,y,t)|/U << 1$ and $|\mathbf{u}(x,y,t)|/U << 1$.

- Normalize length with respect to $c/2$
- Normalize velocity with respect to the freestream U
- Determine a velocity field $\vec{u}(x,y,t)$ such that

$$\nabla \times \vec{u} = \nabla \cdot \vec{u} = 0 \tag{2}$$

- Boundary Conditions: $\vec{u} = \{u, v\}$
  1. $y = 0$, $-1 < x < +1$:
     (a) $v =$ specified
     (b) $v_+ = v_-$
  2. $y = 0$, $x > +1$:
     (a) pressure is continuous, $p_+ = p_-$
     (b) normal velocity is continuous, $v_+ = v_-$
  3. $|\vec{u}| \to 0$ as $(x^2 + y^2)^{\frac{1}{2}} \to \infty$

From the Euler Equations,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) u_\pm = -\frac{1}{\rho}\frac{\partial p_\pm}{\partial x}. \tag{3}$$

Using condition 2,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \Delta u = 0, \tag{4}$$

where $\Delta u = u_+ - u_-$. Therefore,

$$\Delta u = f(x - t), \tag{5}$$

and $(\Delta u)_{T.E.} = f(1 - t)$, is the velocity jump at the trailing.

1.1 Formulation in terms of sectionally analytic functions

$$u - iv = \frac{1}{2\pi i}\int_{-1}^{\infty}\frac{\Delta u - i\Delta v}{x' - z}dx' \tag{6}$$

or

$$u - iv = \frac{1}{2\pi i}\int_{-1}^{1}\frac{\Delta u}{x' - z}dx' + \frac{1}{2\pi i}\int_{1}^{\infty}\frac{f(x-t)}{x' - z}dx' \tag{7}$$

As $z \to x \pm 0,\ -1 < x < +1$,

$$(u - iv)_{\pm} = \pm\frac{1}{2}\Delta u + \frac{1}{2\pi i}⨍_{-1}^{1}\frac{\Delta u}{x' - x}dx' + \frac{1}{2\pi i}\int_{1}^{\infty}\frac{f(x'-t)}{x' - x}dx' \tag{8}$$

Taking the imaginary parts of both sides,

$$v_{\pm} = \frac{1}{2\pi}⨍_{-1}^{1}\frac{\Delta u}{x' - x}dx' + \frac{1}{2\pi}\int_{1}^{\infty}\frac{f(x'-t)}{x' - x}dx' \tag{9}$$

The real parts give $u_{\pm} = \pm\frac{1}{2}\Delta u$, hence, $u_{+} = -u_{-} = \frac{1}{2}\Delta u$.

If we specify the dependence on time $f(x-t)$ will be known. Therefore, we have the following integral equation for the unknown Δu:

$$\frac{1}{\pi}⨍_{-1}^{1}\frac{\Delta u}{x' - x}dx' = 2v - \frac{1}{\pi}\int_{1}^{\infty}\frac{f(x'-t)}{x' - x}dx' \tag{10}$$

The solution to this singular integral equation is not unique. However, if we apply the Kutta condition, i.e., the solution must be finite at the trailing edge, we get

$$\Delta u = -\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\left[⨍_{-1}^{1}\frac{\sqrt{\frac{1+x'}{1-x'}}}{x' - x}\left(2v - \frac{1}{\pi}\int_{1}^{\infty}\frac{f(x''-t)}{x'' - x'}dx''\right)dx'\right]. \tag{11}$$

The double integral can be reduced to a single integral, and we have

$$\Delta u = -\frac{1}{\pi}\sqrt{\frac{1-x}{1+x}}\left[⨍_{-1}^{1}\sqrt{\frac{1+x'}{1-x'}}\frac{2v}{x' - x}dx' - \int_{1}^{\infty}\frac{\sqrt{\frac{x''+1}{x''-1}}f(x''-t)}{x'' - x}dx''\right]. \tag{12}$$

The pressure jump, $\Delta p = p_{+} - p_{-}$, is given by integrating Equation 3.

$$\Delta p = -\left(\Delta u - \Delta u_{T.E.}\right) - \int_{1}^{x}\Delta\dot{u}\,dx' \tag{13}$$

The lift L' is given by

$$L' = -\int_{-1}^{+1}\Delta p\,dx = \int_{-1}^{+1}\Delta u\,dx - 2(\Delta u)_{T.E.} + \int_{-1}^{+1}\int_{1}^{x}\Delta\dot{u}\,dx'\,dx. \tag{14}$$

Integrating the last integral by parts gives

$$L' = \int_{-1}^{+1}\Delta u\,dx - 2(\Delta u)_{T.E.} - \int_{-1}^{+1}(1+x)\Delta\dot{u}\,dx. \tag{15}$$

Evaluation of the integrals is done by substitution of Equation 12 into Equation 15.

$$\int_{-1}^{+1} \Delta u\, dx = -2\int_{-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}v(x',t)dx' + \int_{1}^{\infty} f(x''-t)\left[\sqrt{\frac{x''+1}{x''-1}}-1\right]dx'' \tag{16}$$

$$\int_{-1}^{+1}(1+x)\Delta\dot{u}dx = -2\int_{-1}^{+1}x'\sqrt{\frac{1+x'}{1-x'}}\dot{v}dx' + \int_{1}^{\infty}\dot{f}(x''-t)\left[x''\sqrt{\frac{x''+1}{x''-1}}-(x''+1)\right]dx'' \tag{17}$$

1.2 Application of Kelvin's Theorem

Kelvin's theorem states that the circulation around a circuit moving with the fluid is constant for a barotropic fluid with no viscosity and body forces.

$$0 = \int_{-1}^{\infty}\Delta u\, dx = \int_{-1}^{+1}\Delta u\, dx + \int_{1}^{\infty} f(x''-t)dx'' \tag{18}$$

Substituting Δu from Equation 12 into (18) and using (16)

$$-2\int_{-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}v\, dx' + \int_{1}^{\infty} f(x''-t)\sqrt{\frac{x''+1}{x''-1}}dx'' = 0. \tag{19}$$

This equation will determine the intensity of the vortex shedding in the wake.

1.3 The Lift Formula

Substituting (16 and 17) into (15), gives

$$\begin{aligned} L' \;=\; & -2\int_{-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}v\, dx \\ & +2\int_{-1}^{+1}x'\sqrt{\frac{1+x'}{1-x'}}\dot{v}\, dx' \\ & +\left\{\int_{1}^{\infty} f(x''-t)\left[\sqrt{\frac{x''+1}{x''-1}}-1\right]dx'' - 2(\Delta u)_{T.E.}\right\} \\ & -\int_{1}^{\infty}\dot{f}(x''-t)\left[x''\sqrt{\frac{x''+1}{x''-1}}-(x''+1)\right]dx'' \end{aligned} \tag{20}$$

Noting that

$$\int_{1}^{\infty}(x''+1)\dot{f}(x''-t)dx'' = \int_{1}^{\infty} f(x''-t)dx'' - 2f(1-t), \tag{21}$$

and using **??** to show that

$$\int_{1}^{\infty}\sqrt{\frac{x''+1}{x''-1}}f dx'' - \int_{1}^{\infty}x''\sqrt{\frac{x''+1}{x''-1}}\dot{f}dx'' = \int_{1}^{\infty}\frac{f}{\sqrt{x''^2-1}}dx'' - 2\int_{1}^{1}\sqrt{\frac{1+x'}{1-x'}}dx', \tag{22}$$

gives the following expression for the lift,

$$L' = -2\int_{-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}\,v\,dx' - 2\int_{-1}^{+1}\sqrt{1-x'^2}\,\dot{v}\,dx' + \int_{1}^{\infty}\frac{f}{\sqrt{x''^2-1}}dx''. \quad (23)$$

The first term,

$$L'_{q.s.} = -2\int_{-1}^{+1}\sqrt{\frac{1+x'}{1-x'}}v\,dx'^{1} \quad (25)$$

represents the quasi-steady lift. It is the lift obtained if the velocity, v, is time-independent. For example, if the airfoil is at angle of attack α to the upstream flow, then $v = -\alpha$ and $L'q.s. = 2\pi\alpha$, which is the classical result for the lift coefficient of a thin airfoil. This is also the case when the time variation of v is small, then $\dot{v} \approx 0$ and the vortex shedding in the wake is also negligible, $f(x-t) \approx 0$.

The second term,

$$L'_{a.m.} = -2\int_{-1}^{+1}\sqrt{1-x'^2}\,\dot{v}\,dx'^{2},. \quad (27)$$

is proportional to the acceleration of the velocity $\dot{v}$ and therefore represents the apparent mass lift. This force dominates the response of the airfoil when $\dot{v}$ depends strongly on time.

The last term,

$$L'_w = \int_{1}^{\infty}\frac{f}{\sqrt{x''^2-1}}\,dx'', \quad (28)$$

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[1] In dimensional form, we get,

$$L'_{q.s.} = -2\rho_0 U\int_{-c/2}^{+c/2}\sqrt{\frac{c/2+x'}{c/2-x'}}\,v\,dx' = -2\rho_0 U\int_{0}^{c}\sqrt{\frac{x'}{c-x'}}\,v\,dx'. \quad (25)$$

Note that

$$\begin{aligned}\int_{-c/2}^{+c/2}\sqrt{\frac{c/2+x'}{c/2-x'}}\,dx' &= \pi\frac{c}{2},\\ \int_{0}^{c}\sqrt{\frac{x'}{c-x'}}\,dx' &= \pi\frac{c}{2}.\end{aligned}$$

[2] In dimensional form,

$$L'_{a.m.} = -2\rho_0(\frac{c}{2})^2\int_{-1}^{+1}\sqrt{1-x'^2}\,\dot{v}\,dx' = -2\rho_0\int_{0}^{c}\sqrt{x'(c-x')}\,\dot{v}\,dx'. \quad (27)$$

Note that

$$\begin{aligned}\int_{-1}^{+1}\sqrt{1-x'^2}\,dx' &= \frac{\pi}{2},\\ \int_{0}^{c}\sqrt{x'(c-x')}\,dx' &= \frac{\pi}{8}c^2.\end{aligned}$$

represents the wake shedding contribution to the lift. Thus the lift can be written as the sum of three terms representing respectively, the quasi-steady lift, the apparent mass lift, and the wake induced lift,

$$L' = L'_{q.s.} + L'_{a.m.} + L'_{w}. \tag{29}$$

This result was first derived by von Karmen and Sears in 1938.

1.4 The Moment Formula

An elementary force **dF** produces a moment with respect to the origin O, $\mathbf{dM}/O = x\mathbf{dF}$. The expression for the total moment is

$$\mathbf{M}/O = -\int_{-1}^{+1} x\Delta p dx. \tag{30}$$

Substituting Δp from 13 gives

$$\mathbf{M}/O = \int_{-1}^{+1} x\Delta u\, dx + \int -1^{+1} x \int_{1}^{x} \Delta\dot{u} dx'\, dx. \tag{31}$$

Integrating the last term by parts,

$$\mathbf{M}/O = \int_{-1}^{+1} x\Delta u\, dx + \frac{1}{2}\int_{-1}^{+1} (1-x^2)\Delta\dot{u}\, dx. \tag{32}$$

After considerable reduction, we obtain

$$\mathbf{M}/O = 2\int_{-1}^{+1} \sqrt{1-x^2}\, v\, dx - \int_{-1}^{+1} x\sqrt{1-x^2}\, \dot{v}\, dx - \frac{1}{2}\int_{1}^{\infty} \frac{f}{\sqrt{x^2-1}} dx. \tag{33}$$

As for the lift formula, the moment is the sum of three terms representing the quasi-steady moment, the apparent mass moment and the wake induced moment.

1.5 Harmonic Time Dependence

Equations (23 and 58)determine the lift and moment of an airfoil in terms of the velocity v and the distribution of the vortex shedding in the wake $f(x-t)$. The velocity v is specified but $f(x-t)$ must be determined from the time history of the motion. In many instances the airfoil is undergoing periodic oscillations or subject to periodic gust excitations. For these cases the velocity v is a periodic function of time and it is possible to consider the airfoil response to a single harmonic component $v = \tilde{v}e^{-i\omega t}$. In the general case it is also convenient to consider the Fourier transform

$$v(x,t) = \int_{-\infty}^{+\infty} \tilde{v}(x,\omega)e^{-i\omega t}\, d\omega, \tag{34}$$

and to calculate the response to every Fourier component $\tilde{v}e^{-i\omega t}$. This approach has the advantage of immediately determining the expression for the wake vortex distribution

$$f(x-t) = c_o e^{i(\omega(x-t))}, \tag{35}$$

where c_o is constant which can be readily determined from the Kelvin theorem condition by substituting (35) into (19),

$$c_o e^{-i\omega t} = \frac{2\int_{-1}^{+1}\sqrt{\frac{1+x}{1-x}}\, v\, dx}{\int_1^{\infty}\sqrt{\frac{x+1}{x-1}}\, e^{i\omega x}\, dx}. \tag{36}$$

The numerator in (36) is $-L'_{q.s.}$ and the integral in the denominator can be expressed in terms of Hankel functions, giving

$$c_o e^{-i\omega t} = \frac{-L'_{q.s.}}{i\frac{\pi}{2}H_+(\omega)}, \tag{37}$$

where $H_+(\omega) = H_0^{(1)}(\omega) + iH_1^{(1)}(\omega)$. The wake-induced lift, L'_w, can also be evaluated by substituting (35) into (28),

$$L'_w = -L_{q.s.}\frac{H_0^{(1)}(\omega)}{H_+(\omega)}. \tag{38}$$

Substituting (38) into (23) gives

$$L' = L'_{q.s.}C(\omega) + L'_{a.m.}, \tag{39}$$

where

$$C(\omega) = \frac{i\, H_1^{(1)}(\omega)}{H_+(\omega)} \tag{40}$$

is the complex conjugate of the Theodorsen function[3]. It can be readily shown that as $\omega \to 0, C(\omega) \to 1$, and as $\omega \to \infty, C(\omega) \to 0.5$.
Similarly, the expression for the moment becomes

$$M/O = 2\int_{-1}^{+1}\sqrt{1-x^2}\, v\, dx - \int_{-1}^{+1} x\sqrt{1-x^2}\, \dot{v}\, dx + \frac{1}{2}L'_{q.s.}\left(1 - C(\omega)\right) \tag{41}$$

In what follows we apply the general formulas derived for the lift and moment to special cases. We first consider the cases of translatory and rotational oscillations as models for bending and torsional oscillations. We then study the case for a harmonic transverse gust impinging on the airfoil. Closed form analytical expressions for the lift and moments will be derived for these fundamental cases. The results will also be used to study the fundamental problems of a step change in the angle of attack and that of a sharp edged gust.

1.5.1 Translatory Oscillations

For a translatory oscillation the airfoil normal velocity v is independent of x and has the form $v = \overline{v}e^{-i\omega t}$, where $\overline{v}$, the magnitude of the oscillation, is constant. Using this expression for v in the various lift and moment formulas gives

[3]Theodorsen considered a time dependence of the form $exp(i\omega t)$.

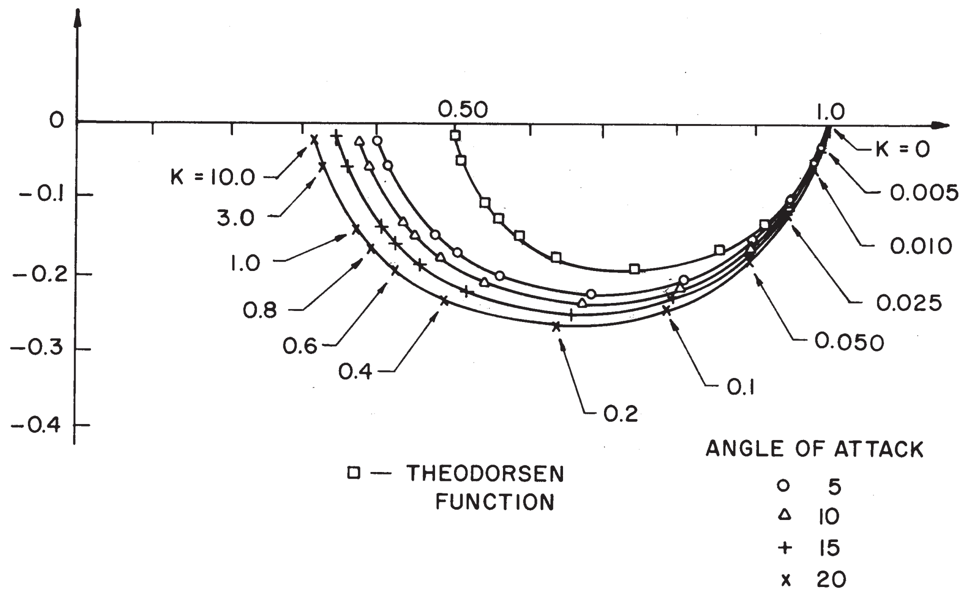

Figure 1: Vector diagram of the real and imaginary parts of the Theodorsen function as the reduced frequency increases from 0 to ∞.

$$L'_{q.s.} = -\pi\rho c U v, \tag{42}$$

$$L'_{a.m.} = -\frac{\pi}{4}\rho c^2 \dot{v}, \tag{43}$$

$$L'_{w} = -L'_{q.s.}\left(1 - C(\omega)\right), \tag{44}$$

$$L' = L'_{q.s.}C(\omega) - \frac{\pi}{4}\rho c^2 \dot{v} \tag{45}$$

$$M_{q.s.}/O = -\frac{c}{4}L'_{q.s}, \tag{46}$$

$$M_{a.m.}/O = 0, \tag{47}$$

$$M_{w}/O = -\frac{c}{4}L'_{w}, \tag{48}$$

$$M/O = -\frac{c}{4}L'_{q.s.}C(\omega) \tag{49}$$

These results show that the center of pressure for both the the quasi-steady lift and the wake-induced lift is located at the quarter chord, and that the center of pressure for the apparent mass lift is located at the origin.

1.5.2 Rotational Oscillations

For an airfoil in rotational oscillation, the airfoil surface is $y - \alpha x = 0$, where $\alpha = \overline{\alpha} exp(-i\omega t)$ is the airfoil angle with the x-axis, and $\overline{\alpha}$,the angular magnitude of the oscillation, is constant. The expression for the normal velocity is obtained by taking $D_0/Dt(y - \alpha)x = 0$, yielding $v = \dot{\alpha}x + \alpha U$. It is seen that the velocity is the sum of two terms. The second term, αU, is independent of x and represents a translatory oscillation whose lift and moment are given by (42 - 49). Thus, we need only to calculate the expressions for the lift and moment in response to the first term $\dot{\alpha}x$ by substituting this expression for v in the various lift and moment formulas. This gives

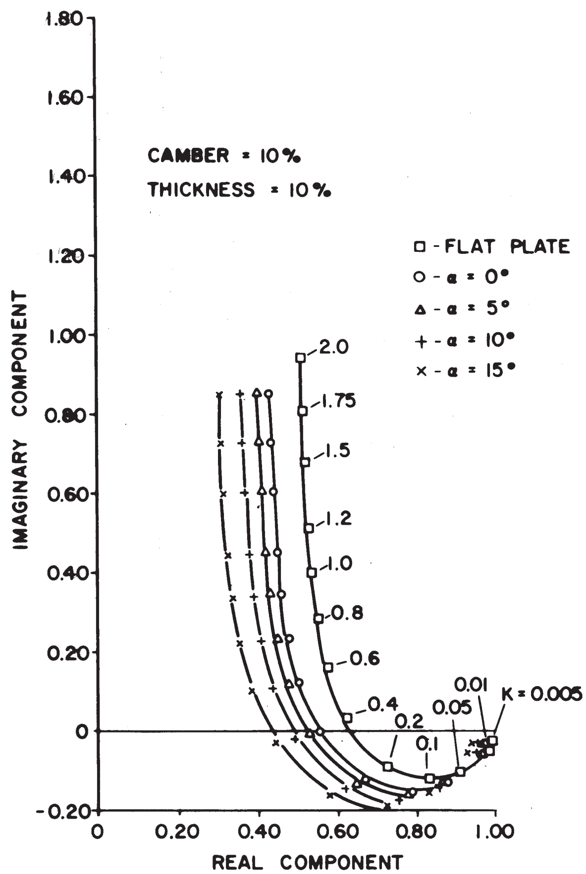

Figure 2: Vector diagram of the real and imaginary parts of the lift coefficient of an airfoil in translatory oscillation.

$$L'_{q.s.} = -\frac{\pi}{4}\rho c^2 U \dot{\alpha}, \quad (50)$$

$$L'_{a.m.} = 0, \quad (51)$$

$$L'_w = -L'_{q.s.}\left(1 - C(\omega)\right), \quad (52)$$

$$
\begin{aligned}
L' &= L'_{q.s.} C(\omega) && (53)\\
M_{q.s.}/O &= 0, && (54)\\
M_{a.m.}/O &= -\frac{\pi}{128}\rho c^4 \ddot{\alpha}, && (55)\\
M_w/O &= -\frac{c}{4} L'_w, && (56)\\
M/O &= -\frac{c}{4} L'_{q.s.} (1 - C(\omega)) - \frac{\pi}{128}\rho c^4 \ddot{\alpha}. && (57)
\end{aligned}
$$

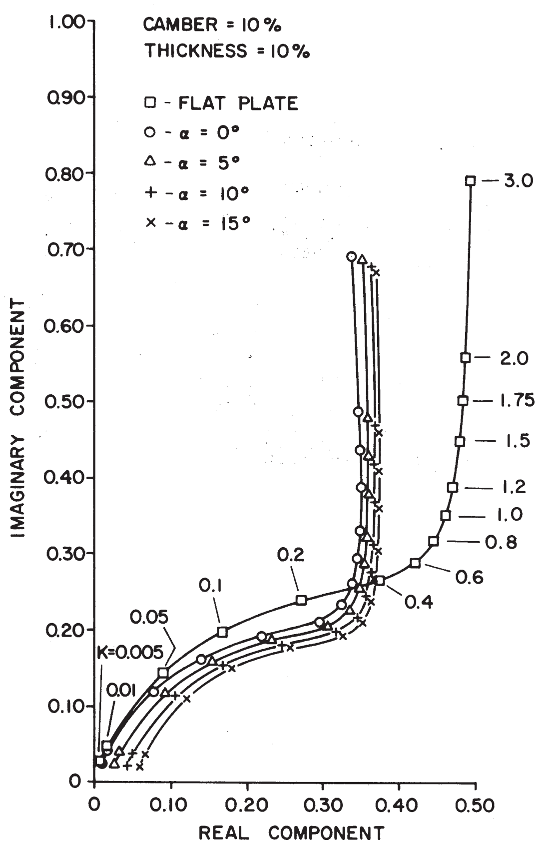

Figure 3: Vector diagram of the real and imaginary parts of the lift coefficient of an airfoil in rotational oscillation.

1.5.3 Translatory and Rotational Oscillations

For an airfoil in translatory and rotational oscillation hinged at the point O', the airfoil surface is $y - \alpha(x - x_{O'}) = 0$, $v = \dot{\alpha}(x - x_{O'}) + \alpha U$. This is equivalent to the sum of a translatory oscillation, $\alpha U - \dot{\alpha} x_{O'}$, whose lift and moment are given by (42 - 49), and a rotaional oscillation about the origin whose lift and moment are given by (50 - 57). Note that the expression for the moment with respect to O' is

$$M/O' = M/O - x_{O'}L'. \tag{58}$$

1.5.4 Step Change in Angle of Attack

This problem examines the time evolution of the lift of an airfoil as the angle of attack α is suddenly increased from zero to α_o, i.e.,

$$\alpha = \begin{cases} 0, & t < 0, \\ \alpha_o, & t > 0. \end{cases}$$

This can be written in terms of the unit step function

$$y + u(t)\alpha_o x = 0, \tag{59}$$

where

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

The normal velocity is then given by

$$v = -u(t)\alpha_o U - \delta(t)\alpha_o x, \tag{60}$$

where $\delta(t)$ is the unit impulse or Dirac function and $\dot{u}(t) = \delta(t)$. Define the unit step and Dirac functions as

$$u(t) = \frac{1}{2} + \frac{1}{2\pi i}\oint_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega} d\omega, \tag{61}$$

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\, d\omega. \tag{62}$$

or,

$$u(t) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \frac{sin\omega t}{\omega} d\omega, \tag{63}$$

$$\delta(t) = \frac{1}{\pi}\int_0^{+\infty} cos\omega t\, d\omega. \tag{64}$$

As in previous results, we calculate the lift in two stages. We first consider the lift produced by the translatory part -$u(t)\alpha_o U$. The constant term in (63) gives a steady lift of $\pi\rho c U^2\alpha_o/2$. The circulatory part of the lift produced by $(\alpha_o U e^{-i\omega t})/(\pi\omega)$ is

$$L'_C = -\frac{\rho c U^2 \alpha_o C(\omega)}{(\omega)}\, e^{-i\omega t} \tag{65}$$

.

Hence, the lift produced by $-(\alpha_o U sin\omega t)/(\pi\omega)$ is the imaginary part of (65). The total circulatory lift can be obtained by integration,

$$L'_C = \Big(\pi\rho c U^2\alpha_o\Big)\left[\frac{1}{2} - \int_0^{\infty} \frac{Im([e^{-i\omega t}C(\omega)]}{\omega}\, d\omega.\right] \tag{66}$$

The non-dimensional form of L'_C

$$\phi(t) = \frac{1}{2} - \int_0^\infty \frac{Im\left[e^{-i\omega t)} C(\omega)\right]}{\omega} d\omega. \tag{67}$$

is the Wagner function. By writing $C(\omega)$ in terms of its real and imaginary parts of $C(\omega) = Re\left[C(\omega)\right] + iIm\left[C(\omega)\right],$

$$\phi(t) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im(C)cos\omega t}{\omega} d\omega + \frac{1}{\pi}\int_0^\infty \frac{Re(C)sin\omega t}{\omega} d\omega. \tag{68}$$

For $t < 0$, the lift must be zero, therefore

$$\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im(C)cos\omega t}{\omega} d\omega = \frac{1}{\pi}\int_0^\infty \frac{Re(C)sin\omega|t|}{\omega} d\omega. \tag{69}$$

The wagner function can then be written as

$$\phi(t) = \frac{2}{\pi}\int_0^\infty \frac{Re(C)sin\omega t}{\omega} d\omega. \tag{70}$$

The limits for ϕ can be obtained as follows. We first note that for $t > 0$,

$$\frac{2}{\pi}\int_0^\infty \frac{sin\omega t}{\omega} d\omega = 1. \tag{71}$$

For $t > 0$, as $t \to 0$ the integrand of (70) vanishes for small and finite values of ω; only large values of ω contribute to (70). Since for $\omega \to \infty$, $C(\omega) \to 0.5$ and using (71), we get $\lim_{t\to 0}\phi(t) = 0.5$. For $t \to \infty$, only vanishing values of ω contributes to (70). Since for $\omega \to 0$, $C(\omega) \to 1$ and using (71), we get $\lim_{t\to\infty}\phi(t) = 1$.
We give the following two approximations for the wagner function,

$$\begin{aligned} \phi(t) &= 1 - 0.165e^{-0.0455t} - 0.335e^{-0.3t} \\ \phi(t) &\sim \frac{t+2}{t+4} \end{aligned}$$

1.5.5 Airfoil in a Gust

We consider a uniform flow U in the x direction with transverse sinusoidal gust in the y direction as shown in figure (5). Thus the unsteady velocity field is of the form

$$\vec{u} = a_2\vec{j}e^{i(k_1x-\omega t)} + \vec{u}, \tag{72}$$

where $k_1 = (\omega c)/(2U)$. Since we non-dimensionalize length and velocity with respect to $c/2$ and U, respectively, we get in non-dimensiosnal form $k_1 = \omega$. At the surface of the airfoil, the normal velocity must vanish,

$$v = -a_2 e^{ik_1x-\omega t)} \text{ for } y = 0, -1 < x < +1. \tag{73}$$

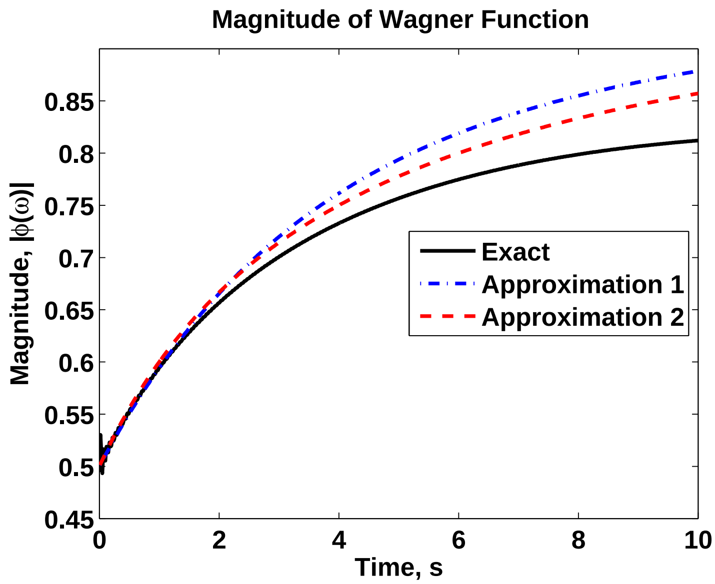

Figure 4: The Wagner function.

Substituting (73) into (39) and noting that

$$\int_{-1}^{+1} \sqrt{\frac{1+x}{1-x}} e^{ik_1 x} = \pi J_+(k_1),$$

$$\int_{-1}^{+1} \sqrt{1-x^2} e^{ik_1 x} = \pi \frac{J_1(k_1)}{k_1},$$

where we have put $J_+ = J_0 + iJ_1$, we get

$$L' = \pi \rho c U a_2 \left[J_+(k_1) C(\omega) - i \frac{\omega}{k_1} J_1(k_1) \right] e^{-i\omega t}. \tag{74}$$

For a gust, $k_1 = \omega$. Using (1) for $C(\omega)$ and noting that $J_+ H_1^{(1)} - J_1 H_+ = -2i/(\pi\omega)$, we obtain the simple expression

$$L' = \pi \rho c U a_2 S(\omega) e^{-i\omega t}, \tag{75}$$

where

$$S(\omega) = \frac{2}{\pi \omega H_+(\omega)}. \tag{76}$$

The function S is the complex conjugate of the well known Sears function[4]. Traditionally, the Sears function is plotted as a vector diagram of the real and imaginary parts versus the reduced frequency as shown in figure(6). The Sears function is widely used in applications. A good approximation for its magnitude is given by

$$|S| = \frac{1}{\sqrt{1 + 2\pi\omega}}. \tag{77}$$

[4]Sears considered a time dependence of the form $exp(i\omega t)$.

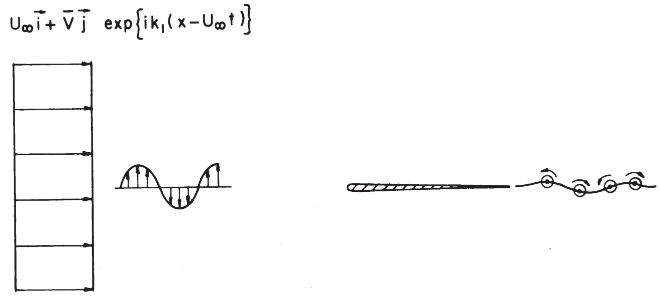

Figure 5: Thin airfoil in a transverse gust.

1.5.6 Sharp-Edged Gust

An incident sharp-edged gust is defined as a traveling vertical gust where

$$v_g = \begin{cases} 0, & x < Ut - c/2 \\ v_o, & x > Ut - c/2 \end{cases} . \tag{78}$$

As for the case of a step change in angle of attack, we use the unit step function to represent the gust velocity. Equation (78) can be written as

$$v_g = v_o u(x - Ut + \frac{c}{2}) \tag{79}$$

Using the expression (63) for the unit step function and following the procedure developed in§1.5.4, gives the expression for the lift

$$L' = \pi \rho c U v_o \left[\frac{1}{2} + \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{S(\omega) e^{-i\omega(t-1)}}{\omega} \right] d\omega, \tag{80}$$

Like the previous case, we can rewrite the lift as

$$L' = \pi \rho c U v_o \psi(t),$$

where

$$\psi(t) = \frac{1}{2} + \frac{1}{2\pi i} \oint_{-\infty}^{+\infty} \frac{S(\omega) e^{-i\omega(t - \frac{c}{2U})}}{\omega} d\omega \tag{81}$$

is the Kussner function. It is convenient to cast the lift in terms of a non-singular integrand

$$\psi(t) = \frac{2}{\pi} \int_0^{+\infty} Re\left[S(\omega) e^{i\omega} \right] \frac{sin\omega t}{\omega} d\omega \tag{82}$$

For $t > 0$, as $t \to 0$ the integrand of (70) vanishes for small and finite values of ω; For large values of ω the Sears function vanishes. Hence, $\lim_{t\to 0} \psi(t) = 0$. For $t \to \infty$, only

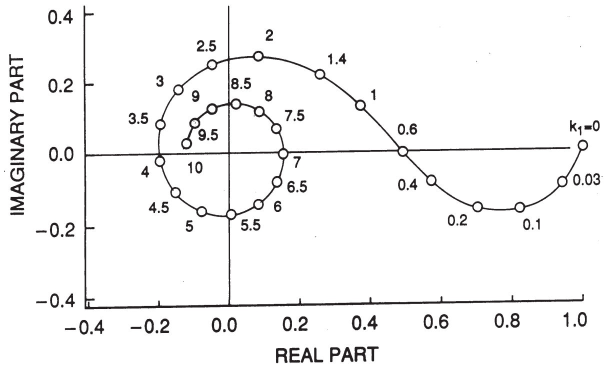

Figure 6: Vector diagram of the real and imaginary parts of the Sears function as the reduced frequency increases from 0 to 10.

vanishing values of ω contributes to (70). Since for $\omega \to 0$, $S(\omega) \to 1$ and using (71), we get $\lim_{t\to\infty} \psi(t) = 1$. The following approximations for ψ were derived

$$\begin{aligned} \psi(t) &= 1 - 0.500e^{-0.130t} - 0.5e^{-t} \\ &\sim \frac{t^2 + t}{t^2 + 2.82t + 0.80} \end{aligned} \tag{83}$$

Figure 8 shows the variation of the Kussner function and a comparison with the two approximations. Remember, the normalization for time is $t^* = \frac{t}{c/(2U)}$.

2 Unsteady Subsonic Flows

The theory developed for incompressible two dimensional unsteady flow paved the way to our understanding of unsteady flow phenomena and its application to aeroelasticity and aeroacoustics. As flight speed increased, it became necessary to account for the effects of Mach number or more complex incident disturbances . A good starting point is to begin with the linearized Euler equations about a mean uniform flow, namely,

$$\frac{D_o}{Dt}\rho + \rho_o \nabla \cdot \vec{u} = 0, \tag{84}$$

and

$$\rho_o \frac{D_o}{Dt}\vec{u} = -\nabla p. \tag{85}$$

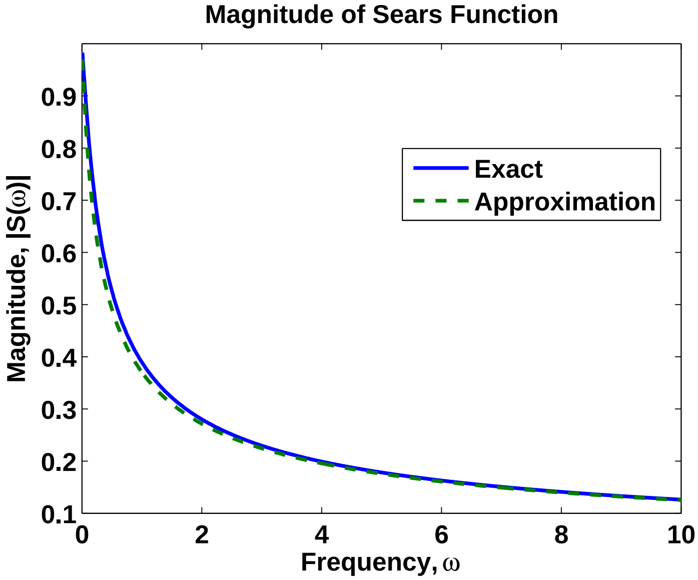

Figure 7: Magnitude of the Sears function.

We further assume the flow to be isentropic and hence $p = c_o^2 \rho'$, where c_0 is the speed of sound. Taking the material derivative of (84 and the divergence of(85 and eliminating the velocity, we get the convected wave equation

$$\left(\frac{1}{c_o^2} \frac{D_o^2}{Dt^2} - \nabla^2 \right) p = 0. \tag{86}$$

Expanding (86) and arranging the terms gives

$$(1 - M^2) \frac{\partial^2 p}{\partial x_1^2} + \frac{\partial^2 p}{\partial x_2^2} + \frac{\partial^2 p}{\partial x_3^2} - 2 \frac{M}{c_o} \frac{\partial^2 p}{\partial x_1 \partial t} - \frac{1}{c_o^2} \frac{\partial^2 p}{\partial t^2} = 0. \tag{87}$$

where M is the Mach number.
We now introduce the Prandtl-Glauert coordinates:

$$\begin{aligned} \tilde{x}_1 &= x_1, \\ \tilde{x}_2 &= x_2 \beta, \\ \tilde{x}_3 &= x_3 \beta, \end{aligned} \tag{88}$$

where $\beta = \sqrt{1 - M^2}$. Equation (87) reduces to

$$\tilde{\nabla}^2 p - 2 \frac{M}{c_o \beta^2} \frac{\partial p}{\partial \tilde{x}_1 \partial t} - \frac{1}{c_o^2 \beta^2} \frac{\partial^2 p}{\partial t^2} = 0, \tag{89}$$

where $\tilde{\nabla}$ is the del operator in the Prandtl-Glauert coordinates.
We now consider the case of time-harmonic dependence of the pressure of the form

$$p = \overline{p}(\mathbf{x}) e^{-i\omega t}. \tag{90}$$

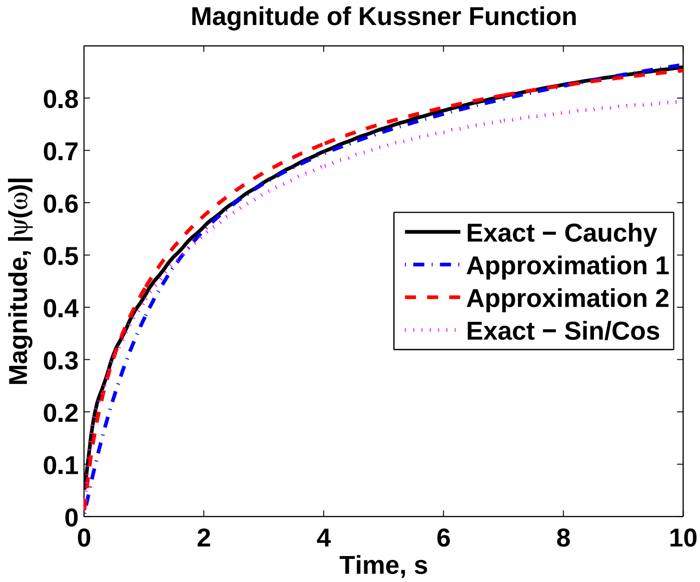

Figure 8: The Kussner function.

Substituting (90) into (86) gives

$$\left[\frac{1}{c_o^2}(\frac{\omega}{c_0}+iM\frac{\partial}{\partial x_1})^2+\nabla^2\right]\tilde{p}=0. \tag{91}$$

or,

$$(1-M^2)\frac{\partial^2\overline{p}}{\partial x_1^2}+\frac{\partial^2\overline{p}}{\partial x_2^2}+\frac{\partial^2\overline{p}}{\partial x_3^2}+2i\omega\frac{M}{c_o}\frac{\partial\overline{p}}{\partial x_1}+\frac{\omega^2}{c_o^2}\frac{\partial^2\overline{p}}{\partial t^2}=0. \tag{92}$$

$$\left[\tilde{\nabla}^2+2i\frac{\omega M}{\beta^2 c_o}\frac{\partial}{\partial x_1}+\frac{\omega^2}{\beta^2 c_o^2}\right]\overline{p}=0, \tag{93}$$

In order to simplify (93), we introduce the new variable

$$\overline{p}=\tilde{p}(\mathbf{x})e^{i\alpha x}.. \tag{94}$$

Note that

$$\frac{\partial\overline{p}}{\partial\tilde{x}_1}=\left(\frac{\partial\tilde{p}}{\partial\tilde{x}_1}+i\alpha\tilde{p}\right)e^{i\alpha\tilde{x}_1} \tag{95}$$

$$\frac{\partial^2\overline{p}}{\partial\tilde{x}_1^2}=\left(\frac{\partial^2\tilde{p}}{\partial\tilde{x}_1^2}+2i\alpha\frac{\partial\tilde{p}}{\partial\tilde{x}_1}-\alpha^2\tilde{p}\right)e^{i\alpha\tilde{x}_1} \tag{96}$$

The governing equation (93) can then be written

$$\tilde{\nabla}^2\tilde{p}+2i\alpha\frac{\partial\tilde{p}}{\partial\tilde{x}_1}-\alpha^2\tilde{p}+\frac{2iM\omega}{c_o\beta^2}\frac{\partial\tilde{p}}{\partial\tilde{x}_1}+\frac{2\omega M\alpha}{c_o\beta^2}\tilde{p}+\frac{\omega^2}{c_o^2\beta^2}\tilde{p}=0. \tag{97}$$

Taking $\alpha=-\frac{M\omega}{c_o\beta^2}$ gives

$$\tilde{\nabla}^2\tilde{p} + \left(\frac{\omega^2}{c_0^2\beta^2} - \frac{\omega^2 M^2}{c_o^2\beta^4} + \frac{2\omega^2 M^2}{c_o^2\beta^4}\right)\tilde{p} = 0, \tag{98}$$

or

$$\left(\tilde{\nabla}^2 + K_\omega^2\right)\tilde{p} = 0, \tag{99}$$

where we have put $K_\omega = \omega/(c_o\beta^2)$. This important result shows that the linearized Euler equations reduce the Helmholtz equation as in acoustics.

Note that the velocity potentioal defined by

$$\mathbf{u} = \nabla\phi \tag{100}$$

also satisfies the convected wave equation (86). Following the same procedure which led to (99) for the pressure and writing $\phi = \tilde{\phi} exp\left[-i\left(K_\omega M\tilde{x}_1 + \omega t\right)\right]$, we get

$$\left(\tilde{\nabla}^2 + K_\omega^2\right)\tilde{\phi} = 0. \tag{101}$$

2.1 Fundamental Solution

The Green's function is solution to

$$\left[\frac{1}{c_o^2}\left(\frac{\omega}{c_0} + iM\frac{\partial}{\partial x_1}\right)^2 + \nabla^2\right] p = -4\pi\delta(\mathbf{x} - \mathbf{x_0}), \tag{102}$$

where $\mathbf{x}$ is the observation point and $\mathbf{x}_0$ is the source point. Expanding (102), dividing both sides by β^2 and noting that $\delta(\mathbf{x} - \mathbf{x_0}) = \delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_\mathbf{0})/\beta^2$, we get

$$\left(\tilde{\nabla}^2 + K_\omega^2\right)\tilde{p} = -4\pi\delta(\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_\mathbf{0})e^{iK_\omega M\tilde{x}_1}, \tag{103}$$

The solution to (103) is

$$\tilde{p} = \frac{e^{iK_\omega(\tilde{r} + M\tilde{x}_{01})}}{\tilde{r}}, \tag{104}$$

where $\tilde{r} = |\tilde{\mathbf{x}} - \tilde{\mathbf{x}}_0|$. The expression for p can be readily obtained as

$$p = \frac{e^{i\{K_\omega[\tilde{r} - M(\tilde{x}_1 - \tilde{x}_{01})] - \omega t\}}}{\tilde{r}}. \tag{105}$$

Or in terms of the physical variables,

$$p = \frac{e^{i\left\{K_\omega r\left[\left(1 - M^2\sin^2\theta\right)^{\frac{1}{2}} - M\cos\theta\right] - \omega t\right\}}}{r\left(1 - M^2\sin^2\theta\right)^{\frac{1}{2}}}, \tag{106}$$

where $r = |\mathbf{x} - \mathbf{x}$ and $\theta = sin^{-1}\sqrt{x_2^2 + x_2^2}/r$ is the angle between the observation point direction and the x_1-axis. For $M = 0$, (106) reduces to the expression for the pressure from a source without mean motion. The effect of the Mach number is to distort the pressure magnitude and phase which become dependent on the angle θ.

2.2 Subsonic Two-Dimensional Unsteady Airfoil Theory

Consider a thin two-dimensional airfoil of infinite span and chord length c. The airfoil may have a small motion about its mean position. Let the x_1 axis be aligned with the airfoil mean position and centered at its midchord O. The flow upstream of the airfoil may have small nonuniformities but its mean velocity $\mathbf{U}$ is uniform and in the x_1-direction. The x_2 axis is in the direction of the airfoil span and the x_3 axis is perpendicular to the airfoil in the upward direction. In what follows we non-dimensionalize length with respect to $c/2$. The velocity field can be written as

$$\mathbf{V}(x, y, t) = \mathbf{U} + \mathbf{u}_\infty(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t), \tag{107}$$

where $\mathbf{u}_\infty(\mathbf{x}, t)$ represents the velocity of the imposed upstream nonuniformities, and $\mathbf{u}(\mathbf{x}, t)$ stands for the perturbation velocity generated by the presence of the airfoil. The assumption of small airfoil motion and small upstream nonuniformities imply that $|\mathbf{u}_\infty(\mathbf{x}, t)|/U << 1$ and $|\mathbf{u}(\mathbf{x}, t)|/U << 1$.

We introduce the potential function

$$\mathbf{u}(\mathbf{x}, t) = \nabla\phi. \tag{108}$$

Then the unsteady pressure

$$p = -\rho_0 \frac{D_0}{Dt}\phi. \tag{109}$$

We introduce the reduced frequency

$$\omega^* = \frac{\omega c}{2U} \tag{110}$$

and the nondimensional form of K_ω

$$K_1 = \frac{\omega^* M}{\beta^2} \tag{111}$$

The boundary conditions are as follows

- The pressure p vanishes at infinity.
- The vertical velocity $\partial\phi/\partial x_3$is specified at $-c/2 < x_1 < c/2,\;\; x_2 = 0$.
- p is continuous in the wake.

We consider time-harmonic dependence. In addition, for two-dimensional airfoils of infinite span it is always possible to consider a Fourier expansion in the span direction and to consider a single component of the form $exp(ik_2x_2)$. This allows us to factor out the time and the x_2 dependence by writing

$$\phi = (\frac{c}{2}u_0)\varphi(\tilde{x}_1, x_3)exp\left[i\left(k_2x_2 - K_1M\tilde{x}_1 - \omega t\right)\right], \tag{112}$$

where u_0 is a representative value for the upwash $u_3 = u_0u_3^*\,exp[i(K_2\tilde{x}_2 - K_1M\tilde{x}_1 - \omega t)]$. The equation for φ can be readily obtained from (101), giving

$$\left(\tilde{\nabla}^2 + K^2\right)\varphi = 0. \tag{113}$$

Here, $\tilde{\nabla}^2$ is the two-dimensional Laplacian operator in the Prandtl-Glauert plane, i.e., $\tilde{\nabla}^2 = \frac{\partial^2}{\partial \tilde{x}_1^2} + \frac{\partial^2}{\partial \tilde{x}_3^2}$ and we have put $K^2 = K_\omega^2 - K_2^2$ with $K_2 = k_2/\beta$.

This reduces the boundary-value problem for the unsteady infinite span airfoil in subsonic flow to that of a two-dimensional boundary-value problem where the governing equation is the two-dimensional Helmholtz equation. The expression for the pressure is given by

$$\begin{aligned} p(\mathbf{x}, t) &= (\rho_0 U u_0)\tilde{p}(\tilde{x}_1, \tilde{x}_3)e^{i(k_2 x_2 - \omega t)}, && (114)\\ \tilde{p}(\tilde{x}_1, \tilde{x}_3) &= (i\frac{\omega^*}{\beta^2}\varphi - \frac{\partial\varphi}{\partial\tilde{x}_1})e^{-iK_1 M \tilde{x}_1}. && (115)\end{aligned}$$

The boundary conditions are

$$\begin{aligned} \frac{\partial\varphi}{\partial\tilde{x}_3} &= \frac{1}{\beta}e^{iK_\omega M\tilde{x}_1}u_3^* && (116)\\ i\frac{\omega^*}{\beta^2}\varphi - \frac{\partial\varphi}{\partial\tilde{x}_1} &= 0 \quad \text{for } \tilde{x}_1 > 1, && (117)\\ \varphi &= 0 \quad \text{as } |\mathbf{x}| \to \infty. && (118)\end{aligned}$$

Equations (113, 116–118) state that aerodynamic problem of an infinite plate vibrating or subject to a three-dimensional gust can be formulated in terms of a two-dimensional problem. The plate vibratory motion must be expanded in terms of its Fourier components in the x_2 direction, i.e., $exp(ik_2x_2)$.

2.3 Integral Equation Formulation

The elementary solution

$$\varphi(\alpha) = e^{i\alpha\tilde{x}_1 - \sqrt{\alpha^2 - K^2}|\tilde{x}_3|},$$

satisfyies the Helmholtz equation (113) for *any* α. Hence, the general solution is given by

$$\varphi = \pm\int_{-\infty}^{+\infty} g(\alpha)e^{i\alpha\tilde{x}_1 - \sqrt{\alpha^2 - K^2}|\tilde{x}_3|}d\alpha. \tag{119}$$

Note that $g(\alpha)$ is unknown. Taking the derivative with respect to $\tilde{x}_3$ gives

$$\frac{\partial\varphi}{\partial\tilde{x}_3} = -\int_{-\infty}^{+\infty}\sqrt{\alpha^2 - K^2}g(\alpha)\,e^{i\alpha\tilde{x}_1 - \sqrt{\alpha^2 - K^2}|\tilde{x}_3|}\,d\alpha. \tag{120}$$

Now along the body, $\tilde{x}_3 = 0$, the equation for $\Delta\varphi$ and $\Delta\tilde{p}$ are

$$\Delta\varphi = 2\int_{-\infty}^{+\infty} g(\alpha)e^{i\alpha\tilde{x}_1}d\alpha. \tag{121}$$

$$\Delta\tilde{p} = 2ie^{-iK_1M\tilde{x}_1}\int_{-\infty}^{+\infty} f(\alpha)e^{i\alpha\tilde{x}_1}d\alpha, \tag{122}$$

where we have set

$$f(\alpha) = (\frac{\omega^*}{\beta^2} - \alpha)g(\alpha) \tag{123}$$

This looks like a Fourier transform. Therefore, can invert it to solve for $f(\alpha)$:

$$f(\alpha) = \frac{-i}{4\pi}\int_{-1}^{+1} \Delta\tilde{p}e^{-i(\alpha-K_1M)\tilde{x}_1}d\tilde{x}_1. \tag{124}$$

Note that the limits of integration collapsed to this region because outside of $(-1,+1)$, $\Delta p \to 0$. Using (120, 122, 123), we get

$$u_3^*(\tilde{x}_1) = \frac{i\beta}{4\pi}e^{-iK_1M\tilde{x}_1}\oint_{-\infty}^{+\infty} \frac{\sqrt{\alpha^2-K^2}}{\frac{\omega^*}{\beta^2}-\alpha}e^{i\alpha\tilde{x}_1}\int_{-1}^{+1} \Delta\tilde{p}\, e^{-i(\alpha-K_\omega M)\tilde{x}_1'}d\tilde{x}_1'\, d\alpha \tag{125}$$

Or,

$$u_3^*(\tilde{x}_1) = \int_{-1}^{+1} \Delta\tilde{p}\,\mathcal{K}(\tilde{x}_1-\tilde{x}_1')\, d\tilde{x}_1', \tag{126}$$

where

$$\mathcal{K}(\xi) = \frac{i\beta}{4\pi}\oint_{-\infty}^{+\infty} \frac{\sqrt{\alpha^2-K^2}}{\frac{\omega^*}{\beta^2}-\alpha}e^{i(\alpha-K_1M)\xi}\, d\alpha. \tag{127}$$

The solution is a series of the form

$$\Delta\tilde{p} = \frac{1}{2}A_o\cot\left(\frac{\theta}{2}\right) + \sum_{m=1}^{\infty} A_m\sin(m\theta), \tag{128}$$

where $\cos\theta = -x$.

It is remarkable to note that equation (125) can be cast in the form

$$U_3(\tilde{x}_1) = \frac{i}{4\pi}\oint_{-\infty}^{+\infty} \frac{\sqrt{\alpha^2-K^2}}{\frac{\omega^*}{\beta^2}-\alpha}e^{i\alpha\tilde{x}_1}\int_{-1}^{+1} \Delta P e^{-i\alpha\tilde{x}_1'}\, d\tilde{x}_1'\, d\alpha, \tag{129}$$

where

$$U_3(\tilde{x}_1) = \frac{1}{\beta} u_3^*(\tilde{x}_1) e^{iK_1 M \tilde{x}_1} \quad (130)$$

$$\Delta P = \Delta \tilde{p} e^{iK_1 M \tilde{x}_1'}. \quad (131)$$

Therefore for the same parameters K and ω^*/β^2 ΔP is only function of the modified upwash U_3.

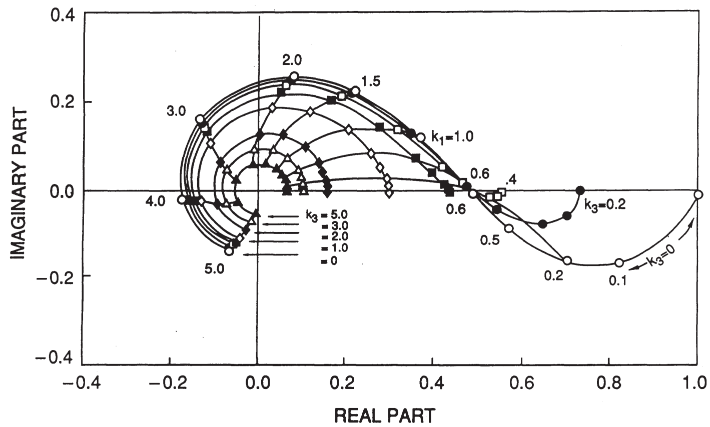

Figure 9: Vector diagram showing the real and imaginary parts of the response function $S(k_1, k_3, M = 0)$ versus k_1 for a transverse gust at various M. Lines of constant k_1 are shown for $k_1 = 0.2, 0.6, 1.0, 1.5, 2.0, 3.0, 4.0, 5.0$.

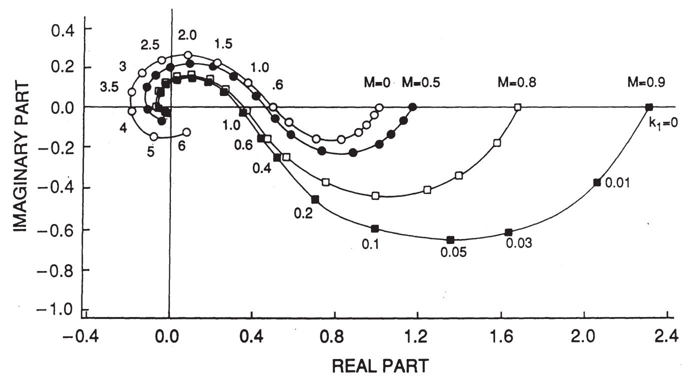

Figure 10: Vector diagram showing the real and imaginary parts of the response function $S(k_1, 0, M)$ versus k_1 for an oblique gust at various k_3.

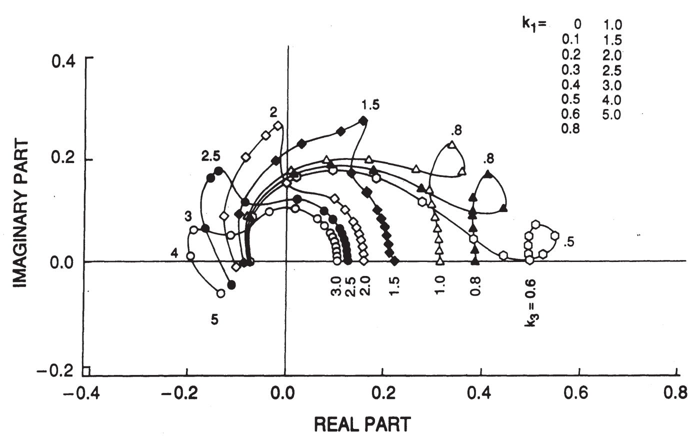

Figure 11: Vector diagram showing the real and imaginary parts of the response function $S(k_1, k_3, M = 0.8)$ versus k_1 for an oblique gust at various k_3.