The Acoustic Problem for a Linear Cascade

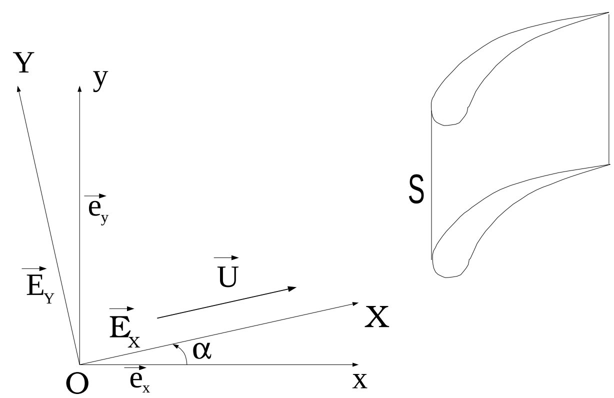

Figure 1: Linear Cascade in a Uniform Flow

Consider a linear cascade of blades with spacing S. Let (x,y,z) be a coordinate system with the axis x in the direction of the machine axis and z in the span direction. The upstream velocity $\vec{U} = U\vec{e}_x$ makes the angle α with the x axis. The spacing is defined by,

$$\vec{S} = S\vec{e}_y = S\left(\sin\alpha\vec{E}_x + \cos\alpha\vec{E}_y\right) \tag{1}$$

A plane wave,

$$\phi = e^{i(\vec{k}\cdot\vec{x}-\omega t)} \tag{2}$$

is propagating in the $\vec{k}$ direction: $\vec{k} = \{k_x, k_y, k_z\}$ in the e frame.

ϕ must satisfy the wave equation,

$$\left(\frac{1}{c_o^2}\frac{D_o^2}{Dt^2} - \nabla^2\right)\phi = 0 \tag{3}$$

This implies that,

$$\vec{k}^2 - \frac{1}{c_o^2}\left(\vec{k}\cdot\vec{U} - \omega\right)^2 = 0 \tag{4}$$

In addition, we impose a quasiperiodicity condition,

$$k_y S = \sigma - 2n\pi \tag{5}$$

where σ is the interblade phase angle.

Expanding (4),

$$k_x^2(1-M_x^2) + 2k_x M_x\left(\frac{\omega}{c_o} - k_y M_y\right) + k_y^2 + k_z^2 - \left(\frac{\omega}{c_o} - k_y M_y\right)^2 = 0 \tag{6}$$

For a propagating wave k_x must be real,

$$M_x^2\left(\frac{\omega}{c_o} - k_y M_y\right)^2 + (1-M_x^2)\left[\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - k_y^2 - k_z^2\right] > 0 \tag{7}$$

Or,

$$\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - (1-M_x^2)(k_y^2 + k_z^2) > 0 \tag{8}$$

$$k_x = \frac{-M_x\left(\frac{\omega}{c_o} - k_y M_y\right) \pm \sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - (1-M_x^2)(k_y^2+k_z^2)}}{(1-M_x^2)} \tag{9}$$

Note that for a given interblade phase angle σ, k_y is given by (5). Thus, we have propagating modes $k_x^{(n)}$ where n is determined by the inequality (8).

Let us examine the condition for a mode (n) to propagate, to this end we consider two important cases. For the first case we assume σ to be constant and independent of ω as in the case of a flutter. In the second case, we consider the acoustics radiated in response to a gust.

1 σ =constant

a. if $\frac{\omega}{c_o} > k_y M_y$, then

$$\frac{\omega}{c_o} > k_y M_y + \sqrt{(1-M_x^2)(k_y^2+k_z^2)} \tag{10}$$

b. If $\frac{\omega}{c_o} < k_y M_y$, then

$$\frac{\omega}{c_o} < k_y M_y - \sqrt{(1-M_x^2)(k_y^2+k_z^2)} \tag{11}$$

However, $k_y M_y$ is always less than $\sqrt{(1-M_x^2)k_y^2}$ since,

$$k_y^2 M_y^2 < (1-M_x^2)k_y^2 \tag{12}$$

as $\left[1-(M_x^2+M_y^2))\right]k_y^2 > 0$. This implies that the r.h.s. is negative, and that the above condition requires $\frac{\omega}{c_o} < 0$ which is not possible.

Therefore, for σ =constant, the condition for a mode to exist is,

$$\frac{\omega}{c_o} > k_y M_y + \sqrt{(1-M_x^2)(k_y^2+k_z^2)} \tag{13}$$

or substituting (5) into (13),

$$\frac{\omega S}{c_o} > (\sigma - 2n\pi)M_y + \sqrt{(1-M_x^2)\left[(\sigma-2n\pi)^2 + k_z^2 S^2\right]} \tag{14}$$

Note that the r.h.s. is always positive for any n. For $n < 0$, the r.h.s. is monotonically increasing with n. So, the modes are ordered : -1 cuts on before -2 and n before $n-1$.

For $n > 0$, and $k_z = 0$, the modes are similarly ordered by n; n cuts on before $n-1$. If $k_z \neq 0$, it is possible depending on σ and k_z to have a mode at higher n cuts on before a lower one. However, for larger n, the modes will appear in the n order.

2 For a gust $\vec{a}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$$\sigma = \vec{k}\cdot\vec{S} \tag{15}$$

For large structure turbomachinery, $k_1 = \frac{\omega}{U}$, $k_2 = k_1 \cot\mu$. Thus $\sigma = \kappa\omega\frac{S}{U}$, where $\kappa = \frac{1}{S}(S_1 + \cot\mu S_2) = (\sin\alpha + \cot\mu\cos\alpha)$.

Condition (8) becomes,

$$\left[\frac{\omega S}{c_o} - M_y(\sigma - 2n\pi)\right]^2 - (1-M_x^2)\left[(\sigma-2n\pi)^2 + S^2k_z^2\right] > 0 \tag{16}$$

or,

$$\left[\frac{\omega S}{c_o}(1-\frac{\kappa M_y}{M}) + 2n\pi M_y\right]^2 - (1-M_x^2)\left[(\kappa\frac{\omega S}{U} - 2n\pi)^2 + S^2k_z^2\right] > 0 \tag{17}$$

Let $\tilde{\omega} = \frac{\omega S}{c_o}$,

$$\tilde{\omega}^2\left[(1-\frac{\kappa M_y}{M})^2 - (1-M_x^2)\frac{\kappa^2}{M^2}\right] + 2\tilde{\omega}\left[(1-\frac{\kappa M_y}{M})2n\pi M_y + (1-M_x^2)\frac{2n\pi\kappa}{M}\right] + 4n^2\pi^2M_y^2 - (1-M_x^2)(4n^2\pi^2 + S^2k_z^2) > 0 \tag{18}$$

$$\tilde{\omega}^2\left(1-\frac{\kappa^2\beta^2}{M^2}-\frac{2\kappa M_y}{M}\right)+2\tilde{\omega}(2n\pi)\overbrace{\left(M_y-\kappa M+\frac{\kappa}{M}\right)}^{M_y+\frac{\kappa\beta^2}{M}}$$
$$-4n^2\pi^2\beta^2-(1-M_x^2)S^2k_z^2>0 \tag{19}$$

The roots of the trinomial are $\tilde{\omega}_1 > \tilde{\omega}_2$.

If,

$$\left(1-\frac{\kappa^2\beta^2}{M^2}-\frac{2\kappa M_y}{M}\right)>0 \tag{20}$$

Then $\tilde{\omega}_1 > 0$ and $\tilde{\omega}_2 < 0$ and the acoustic wave will propagate only if $\tilde{\omega} > \tilde{\omega}_1$. k_z will increase $\tilde{\omega}_1$.

If,

$$\left(1-\frac{\kappa^2\beta^2}{M^2}-\frac{2\kappa M_y}{M}\right)<0 \tag{21}$$

Then both $\tilde{\omega}_1 > 0$ and $\tilde{\omega}_2 > 0$. The acoustic mode propagate for $\tilde{\omega}_2 < \tilde{\omega} < \tilde{\omega}_1$. $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are called the cut-off and cut-on frequencies, respectively. k_z will increase $\tilde{\omega}_2$ and reduce $\tilde{\omega}_1$.

Note that for lower M, the coefficient of $\tilde{\omega}^2$ is negative and as a result, a mode n cuts on at $\tilde{\omega} = \tilde{\omega}_2$. However, this same mode will cut-off at $\tilde{\omega} = \tilde{\omega}_1$.

For a flat plate cascade, $\alpha = \chi$ and $\kappa = \frac{\cos\alpha-\mu}{\sin\mu}$.

$\kappa = \frac{\cos\nu}{\sin\mu}$, $\mu = \tan^{-1}\frac{k_1}{k_2}$.

The Acoustic Intensity

$$\vec{I} = \left(\frac{p'}{\rho_o} + \vec{U} \cdot \vec{u}\right)(\rho_o \vec{u} + \rho' \vec{U}) \tag{22}$$

$$\frac{p'}{\rho_o} = -\frac{D_o}{Dt}\phi = i(\omega - \vec{U} \cdot \vec{k})\phi \tag{23}$$

$$\vec{u} = \nabla\phi = i\vec{k}\phi \tag{24}$$

$$\frac{p'}{\rho_o} + \vec{u} \cdot \vec{U} = i\omega\phi \tag{25}$$

whose real part is, $-\omega \sin(\vec{k} \cdot \vec{x} - \omega t)$.

$$\rho_o\left(\vec{u} + \frac{p'}{\rho_o}\vec{U}\right) = i\rho_o\phi\left[\vec{k} + \left(\frac{\omega}{c_o} - \vec{M} \cdot \vec{k}\right)\vec{M}\right] \tag{26}$$

whose real part is, $-\rho_o\left[\vec{k} + \left(\frac{\omega}{c_o} - \vec{M} \cdot \vec{k}\right)\vec{M}\right]\sin(\vec{k} \cdot \vec{x} - \omega t)$.

$$\vec{I} = \rho_o \omega \sin^2(\vec{k} \cdot \vec{x} - \omega t)\left[\vec{k} + \left(\frac{\omega}{c_o} - \vec{M} \cdot \vec{k}\right)\vec{M}\right] \tag{27}$$

The intensity in the x-direction,

$$I_x = \pm\rho_o \omega \sin^2(\vec{k} \cdot \vec{x} - \omega t)\sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - \beta_x^2(k_y^2 + k_z^2)} \tag{28}$$

The $+$ sign corresponds to acoustic energy flowing in the downstream direction and the $-$ to acoustic energy in the upstream direction. Note that at cut-on condition $I_x = 0$.

Thus, the $+$ sign in (9) corresponds to downstream propagating waves, while the $-$ corresponds to upstream waves.

The average energy crossing area $S \times b$

$$\bar{E} = \rho_o \omega \frac{Sb}{2}\sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - \beta_x^2(k_y^2 + k_z^2)} \tag{29}$$

The Group Velocity

First, we note that the phase velocity,

$$C_{ph} = \frac{\omega}{k_x} = \frac{\beta_x^2 \omega}{-M_x\left(\frac{\omega}{c_o} - k_y M_y\right) \pm \sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - \beta_x^2 (k_y^2 + k_z^2)}} \tag{30}$$

The group velocity,

$$C_g = \frac{d\omega}{dk_x} = \frac{1}{\frac{dk_x}{d\omega}} \tag{31}$$

$$\begin{aligned} \frac{1}{C_g} = \frac{dk_x}{d\omega} &= \frac{1}{\beta_x^2}\left[-\frac{M_x}{c_o} \pm \frac{1}{c_o}\frac{\frac{\omega}{c_o} - k_y M_y}{\sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - \beta_x^2 (k_y^2 + k_z^2)}}\right] \\ &= \frac{1}{\beta_x^2 c_o}\left[-M_x \pm \frac{\frac{\omega}{c_o} - k_y M_y}{\sqrt{\left(\frac{\omega}{c_o} - k_y M_y\right)^2 - \beta_x^2 (k_y^2 + k_z^2)}}\right] \end{aligned} \tag{32}$$

The second term in the bracket is > 1, and the first is < 1. Therefore, the sign of C_g is $\pm$ as that of the second term in the bracket. This confirms that the energy flows either upstream or downstream following the $\pm$ in (8).

For upstream waves, $-$, the bracket is > 1, hence $C_g < c_o$. Near cut-on, the square root$\rightarrow 0$ and $C_g \rightarrow 0$.

For downstream waves,

$$\frac{1}{C_g} > \frac{1}{\beta_x^2 c_o}\left[-M_x + 1\right] = \frac{1}{c_o(1 + M_x)} \tag{33}$$

or $C_g < c_o(1 + M_x)$.