1 Fundamental Solutions to the Wave Equation

Physical insight in the sound generation mechanism can be gained by considering simple analytical solutions to the wave equation. One example is to consider acoustic radiation with spherical symmetry about a point $\vec{y} = \{y_i\}$, which without loss of generality can be taken as the origin of coordinates. If t stands for time and $\vec{x} = \{x_i\}$ represent the observation point, such solutions of the wave equation,

$$(\frac{\partial^2}{\partial t^2} - c_o^2 \nabla^2)\phi = 0, \tag{1}$$

will depend only on the $r = |\vec{x} - \vec{y}|$. It is readily shown that in this case (1) can be cast in the form of a one-dimensional wave equation

$$(\frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial r^2})(r\phi) = 0. \tag{2}$$

The general solution to (2) can be written as

$$\phi = \frac{f(t - \frac{r}{c_o})}{r} + \frac{g(t + \frac{r}{c_o})}{r}. \tag{3}$$

The functions f and g are arbitrary functions of the single variables $\tau_\pm = t \pm \frac{r}{c_o}$, respectively. They determine the pattern or the phase variation of the wave, while the factor $1/r$ affects only the wave magnitude and represents the spreading of the wave energy over larger surface as it propagates away from the source. The function $f(t - \frac{r}{c_o})$ represents an outwardly going wave propagating with the speed c_o. The function $g(t + \frac{r}{c_o})$ represents an inwardly propagating wave propagating with the speed c_o.

2 The Pulsating Sphere

Consider a sphere centered at the origin and having a small pulsating motion so that the equation of its surface is

$$r = a(t) = a_0 + a_1(t), \tag{4}$$

where $|a_1(t)| << a_0$. The fluid velocity at the sphere surface is

$$v_r = \frac{dr}{dt} = \dot{a}(t). \tag{5}$$

At the surface of the sphere

$$(\frac{\partial \phi}{\partial r})_a = \dot{a}(t). \tag{6}$$

A Taylor expansion of (6) gives

$$(\frac{\partial \phi}{\partial r})_a = (\frac{\partial \phi}{\partial r})_{a_0} + (a - a_0)(\frac{\partial^2 \phi}{\partial r^2})_{a_0} + \cdots \tag{7}$$

If ω and λ are representative of the frequency and wave length associated with the acoustic wave, then $(a - a_0)(\frac{\partial^2 \phi}{\partial r^2})_{a_0} \approx larger\{\frac{a_1 \omega}{c_0}\dot{a},\ \frac{a_1}{a_0}\dot{a}\}$. Hence, if $\frac{|a_1|\omega}{c_0} << 1$, or equivalently $\frac{a_1}{\lambda} << 1$

, then $|(a-a_0)(\frac{\partial^2\phi}{\partial r^2})_{a_0}| << |\dot{a}|$. This allows us to linearize the boundary condition along the sphere by transferring it to the mean position at a_0,

$$(\frac{\partial\phi}{\partial r})_{a_0} = \dot{a}(t). \tag{8}$$

The velocity potential can be expressed as in (3). Moreover since the sphere pulsating motion is the source of acoustic waves, the principle of causality suggests that $g \equiv 0$. Thus

$$\phi = \frac{f(t-\frac{r}{c_o})}{r}. \tag{9}$$

Applying the condition (8) at the sphere mean location,

$$\frac{\partial\phi}{\partial r} = -\frac{f(t-\frac{a_0}{c_o})}{a_0^2} - \frac{\dot{f}(t-\frac{a_0}{c_o})}{a_0 c_o} = \dot{a}(t) \tag{10}$$

Integration of (10) gives

$$f(t) = -a_0 c_o \int_{-\infty}^{t} \dot{a}(t'+\frac{a_0}{c_0})e^{-\frac{c_o}{a_0}(t-t')}dt'. \tag{11}$$

Note that if T is a representative period of the sphere pulsation, $c_o T/a_0 = \lambda/a_0$, where λ is a representative of the sound wave length. If $\lambda/a_0 >> 1$, then most of the contribution to the integral (11) is when $t' \approx t$. Neglecting terms of $O(a_0/\lambda)$, we get

$$f(t) = -a_0^2\dot{a}(t), \tag{12}$$

and the acoustic field potential function is given by

$$\phi = -\frac{a_0^2\dot{a}(t-\frac{r}{c_o})}{r}. \tag{13}$$

The expression for the acoustic pressure is

$$p' = \rho_0 a_0^2 \frac{\ddot{a}(t-\frac{r}{c_o})}{r} \tag{14}$$

It is convenient to cast (13,14) in terms of the mass flow rate crossing the sphere of radius a_0, $m(t) = 4\pi\rho_0 a_0^2\dot{a}$. $f(t) = -\frac{m}{4\pi\rho_0}$ and

$$\phi = -\frac{m(t-\frac{r}{c_o})}{4\pi\rho_0 r}, \tag{15}$$

$$p' = \frac{\dot{m}(t-\frac{r}{c_o})}{4\pi r}, \tag{16}$$

$$v_r = \frac{1}{4\pi\rho_0}(\frac{\dot{m}(t-\frac{r}{c_0})}{r c_0} + \frac{m(t-\frac{r}{c_0})}{r^2}) \tag{17}$$

This suggests that in the farfield, i.e., $r >> \lambda$, the acoustic pressure and velocity are in phase and the specific acoustic impedance $z = p'/v_r = \rho_0 c_0$.

2.1 Harmonic Motion

If we have a harmonic motion

$$\dot{a} = \bar{v}e^{-i\omega t}, \tag{18}$$

where $\bar{v}$ is the amplitude of the pulsation velocity and ω its frequency. Substituting (18) into (11) and carrying out the integration, we get

$$f(t) = -a_0 c_0 \bar{v} \frac{e^{-i\omega(t+\frac{a_0}{c_0})}}{\frac{c_0}{a_0} - i\omega}. \tag{19}$$

The expressions for the potential function, velocity and the pressure can be readily obtained by substituting (19) into (9),

$$\begin{aligned}
\phi &= -\frac{\bar{m}}{4\pi\rho_0 r\sqrt{1+\tilde{\omega}^2}}\ e^{i(k(r-a_0)+\varphi-\omega t)}, & (20)\\
v_r &= \frac{\bar{m}}{4\pi\rho_0 r\sqrt{1+\tilde{\omega}^2}}(\frac{-ikr+1}{r})\ e^{i(k(r-a_0)+\varphi-\omega t)}, & (21)\\
p' &= -\frac{i\omega\bar{m}}{4\pi r\sqrt{1+\tilde{\omega}^2}}\ e^{i(k(r-a_0)+\varphi-\omega t)}. & (22)
\end{aligned}$$

where we have introduced $\tilde{\omega} = \omega a_0/c_0$, $\varphi = tan^{-1}\tilde{\omega}$, $k = \omega/c_0$, and $\bar{m} = 4\pi a_0^2\bar{v}\rho_0$. Note that the velocity is the sum of two terms. One terms is in-phase with the pressure and at large distance decays as $1/r$. The other term is out of phase with the pressure and decays at large distance as $1/r^2$.

The specific acoustic impedance

$$z = \frac{p'}{v_r} = R + iX = \rho_0 c_0 \frac{kr}{1+k^2r^2}(kr - i), \tag{23}$$

where R is the resistance and X the reactance. In the farfield where $r >> \lambda$ or, $kr >> 1$, the specific acoustic impedance is dominated by R which has the same value as a plane wave, $z = \rho_0 c_0$. However, in the near field the reactance $|X|$ is comparable to R for $kr = O(1)$ and $|X| >> R$ for $kr << 1$.

The expression for the instantaneous acoustic intensity is given by

$$I = \frac{|p'|^2}{R^2+X^2}\left[Rsin^2(k(r-a_0)+\varphi-\omega t) + \frac{X}{2}sin[2(k(r-a_0)+\varphi-\omega t))]\right]. \tag{24}$$

Note $R/(R^2+X^2) = 1/(\rho_0 c_0)$ and $X/(R^2+X^2) = -1/(\rho_0 c_0 kr)$, thus (24) simplifies to

$$I = \frac{|p'|^2}{\rho_0 c_0}\left[sin^2(k(r-a_0)+\varphi-\omega t) - \frac{1}{2kr}sin[2(k(r-a_0)+\varphi-\omega t))]\right]. \tag{25}$$

The term associated with the reactance X results from coupling the pressure with the out-of-phase term of the velocity. Its time average is zero and hence it does not contribute to the propagation of acoustic energy. Only the term associate with the resistance, which is the result of coupling the in-phase velocity with the pressure produces radiated acoustic energy. The average acoustic intensity are,

$$\bar{I} = \frac{\bar{m}^2\omega^2}{32\pi^2\rho_0 c_0(1+\tilde{\omega}^2)r^2}, \tag{26}$$

$$\mathcal{P} = \frac{\bar{m}^2\omega^2}{8\pi\rho_0 c_0(1+\tilde{\omega}^2)}. \tag{27}$$

For a sphere of small radius compared to the wave length, i.e., $a_0 << \lambda$, the acoustic impedance $|z| << \rho_0 c_0$ in the near field. Moreover, since $R << X$, the impedance is strongly reactive and the surrounding fluid acts mainly as an inertial mass. The velocity is practically out of phase with the pressure. Thus, a source of small size is an inefficient radiator of acoustic energy.

3 The Simple Source

The limit of the pulsating sphere solution as the sphere radius a_0 becomes very small represents the simple source or *monopole* solution. In this case, the source is characterized by the source mass flow rate

$$m(t) = 4\pi a_0^2 \dot{a}(t),$$

and the exact solution is given by (15). If the source is located at the point $|\vec{y}|$, then

$$\phi = -\frac{m(t-\frac{r}{c_o})}{4\pi\rho_0 r}, \tag{28}$$

where $r = |\vec{x}-\vec{y}|$. Equation (28) states that, at the observation point $\vec{x}$ and time t, the sound signal received was emitted from the source point $\vec{y}$ at the *retarded time* $\tau = t - \frac{r}{c_o}$.

The velocity and pressure are given by

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{4\pi\rho_0}\left[\frac{\dot{m}(t-\frac{r}{c_0})}{rc_0} + \frac{m(t-\frac{r}{c_0})}{r^2}\right] \tag{29}$$

$$p' = -\rho_0\frac{\partial\phi}{\partial t} = \frac{\dot{m}(t-\frac{r}{c_0})}{4\pi r} \tag{30}$$

3.1 Harmonic Sources:

In this case

$$m = \bar{m}e^{-i\omega t} \tag{31}$$

$$\phi = -\frac{\bar{m}}{4\pi\rho_0 r}e^{-i\omega\tau} = -\frac{\bar{m}}{4\pi\rho_0 r}e^{i(kr-\omega t)}, \tag{32}$$

where $\bar{m}$ represent the strength of the source and $k = \omega/c_0$. Similarly, the the velocity expression is given by

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{4\pi\rho_0}\left[\frac{\dot{m}(t)}{rc_0} + \frac{m(t)}{r^2}\right]e^{ikr} \tag{33}$$

or

$$v_r = \frac{\bar{m}}{4\pi\rho_0}\left[\frac{-i\omega}{rc_0} + \frac{1}{r^2}\right]e^{i(kr-\omega t)} \tag{34}$$

Noting that $\frac{\omega}{c_0} = \frac{2\pi}{\lambda}$,

$$v_r = \frac{\bar{m}}{4\pi\rho_0}\left[\frac{-i2\pi}{r\lambda} + \frac{1}{r^2}\right]e^{i(kr-t)} \tag{35}$$

$$p' = -\frac{i\omega\bar{m}}{4\pi r}e^{i(kr-\omega t)}. \tag{36}$$

At large distance, $r \gg \lambda$, the acoustic intensity is given by

$$I = p'v_r = \frac{\bar{m}^2\omega^2}{16\pi^2\rho_o c_0 r^2}\sin^2(\omega(t - \frac{r}{c_0})) \tag{37}$$

and

$$\bar{I} = \frac{1}{2}Re(p'\bar{v}_r) = \frac{\bar{m}^2\omega^2}{32\pi^2\rho_o c_0 r^2} \tag{38}$$

$$\bar{\mathcal{P}} = \frac{\bar{m}^2\omega^2}{8\rho_o\pi c_0} \tag{39}$$

3.2 Simple source distribution:

Suppose we have N sources located at $\vec{y_i}$ with strength m_i, then the principle of superposition states that:

$$\phi = \sum_{i=1}^{N}\phi_i = -\frac{1}{4\pi}\sum_{i=1}^{N}\frac{m_i(t - \frac{r_i}{c_0})}{r_i} \tag{40}$$

where $r_i = |\vec{x} - \vec{y_i}|$.

4 The Dipole:

Consider two sources of equal and opposite strength $\pm m$ located at $\vec{y} \pm \vec{\ell}/2$. The potential of the two sources is

$$\phi = -\frac{1}{4\pi\rho_0}[\frac{m(t-\frac{r_+}{c_0})}{r_+} - \frac{m(t-\frac{r_-}{c_0})}{r_-}], \tag{41}$$

where $r = |\vec{x} - \vec{y}|$ and $r_\pm = |\vec{x} - (\vec{y} \pm \vec{\ell}/2)|$. We further assume $|\vec{\ell}| \ll r$, then

$$\phi = \frac{1}{4\pi\rho_0}\ell_i \frac{\partial}{\partial x_i}\left[\frac{m(t-\frac{r}{c_0})}{r}\right] + \ldots \tag{42}$$

or

$$\phi = \frac{1}{4\pi\rho_0}\nabla \cdot \left[\frac{\vec{\ell}m(t-\frac{r}{c_0})}{r}\right] + \ldots. \tag{43}$$

The pressure is given by

$$p' = -\frac{1}{4\pi}\nabla \cdot \left[\frac{\vec{f}(t-\frac{r}{c_0})}{r}\right] + \ldots, \tag{44}$$

or

$$p' = -\frac{1}{4\pi}\frac{\partial}{\partial x_i}(\frac{f_i(t-\frac{r}{c_0})}{r}), \tag{45}$$

where $\vec{f} = \{f_i\} = \dot{m}\vec{\ell}$. Note that $\vec{f}$ has the dimension of a force. The pressure can be more explicitly expressed as

$$p' = \frac{1}{4\pi}[\frac{1}{rc_0}\frac{\partial f_i(t-\frac{r}{c_0})}{\partial t} + \frac{f_i(t-\frac{r}{c_0})}{r^2}] \cdot \frac{(x_i - y_i)}{r}, \tag{46}$$

We can always assume that dipole is located at a finite distance, i.e., $|\vec{y}|$ is finite. The far field acoustic velocity and pressure are the leading terms of those fields as the observer $\vec{x} \to \infty$,

$$p' = \frac{1}{4\pi c_0|\vec{x}|}\frac{d\vec{f}(t-\frac{r}{c_0})}{dt} \cdot \frac{\vec{x}}{|\vec{x}|}. \tag{47}$$

or

$$p' = \frac{1}{4\pi c_0|\vec{x}|}\frac{df(t-\frac{r}{c_0})}{dt} cos\theta, \tag{48}$$

where θ is the angle between $\vec{f}$ and $\vec{x}$ and $f = |\vec{f}|$. Note that for $|\vec{y}| << |\vec{x}|$, r has the following expansion

$$r = |\vec{x}| - \frac{\vec{y} \cdot \vec{x}}{|\vec{x}|} + O(\frac{1}{|\vec{x}|}). \tag{49}$$

Note that the general expression for the velocity can be obtained by from (43). However, in the farfield, the acoustic velocity reduces to its radial component, v_r, whose expression is simply given in terms of the acoustic pressure

$$v_r = \frac{p'}{\rho_0 c_0}. \tag{50}$$

The acoustic intensity has the following expression

$$\mathbf{I} = p'\mathbf{v}. \tag{51}$$

The average acoustic intensity is then given by

$$\overline{I} = \frac{1}{2}\frac{p'\overline{p'}}{\rho_o c_o.}. \tag{52}$$

Using (48), we get

$$\overline{I} = \frac{\dot{f}(t-\frac{r}{c_0})\overline{\dot{f}(t-\frac{r}{c_0})}cos^2\theta}{32\pi^2\rho_0 c_0^3 r^2}. \tag{53}$$

The total power radiated is obtained by integration of a sphere of radius r, this gives

$$\mathcal{P} = \frac{1}{24\pi}\frac{\dot{f}(t-\frac{r}{c_0})\overline{\dot{f}(t-\frac{r}{c_0})}}{\rho_0 c_0^3}. \tag{54}$$

In many applications, the force representing the dipole is created by the interaction of flow nonuniformities and turbulence interaction with structural components such as wings, fan and compressor blades. The fluctuating pressure along the surface of these structure represent a dipole distribution. The farfield acoustics due to this unsteady pressure can be calculated by summing the contribution of all the dipoles. As an example, we consider a flat plate airfoil in a nonuniform flow composed of a uniform flow V in the x_1 direction and a transverse gust disturbance

$$v = \bar{v}e^{i(k_1 x_1 - \omega t)}, \tag{55}$$

Where $k_1 = \frac{\omega}{V}$. The surface pressure jump is given by

$$\Delta p'_s = -2\rho_0 V^2 \frac{\bar{v}}{V}\sqrt{\frac{\frac{c}{2}-y_1}{\frac{c}{2}+y_1}}S(\omega^*)e^{-i\omega t}, \tag{56}$$

where $\omega^* = \omega c/(2V)$ is the reduced frequency and $S(\omega^*)$ is the Sears function. The elementary force applied on the elementary surface $dA = dy_1 dy_2$ is $df = -\Delta p'_s dA$. The acoustic pressure is given by

$$p' = \frac{1}{4\pi c_0|\vec{x}|}2\rho_0 V^2 \frac{\bar{v}}{V}S(\omega^*)\int_{-b/2}^{+b/2}\int_{-c/2}^{+c/2}\sqrt{\frac{\frac{c}{2}-y_1}{\frac{c}{2}+y_1}}e^{-i\omega(t-\frac{|\vec{x}-\vec{y}|}{c_0})}dy_1 dy_2. \tag{57}$$

For simplicity we use the compact source approximation which assumes that the body size is small compared o the wave length. Here, it implies $\omega^* << 1$ or $c << \lambda$. In this case

$$f = \pi\rho_0 cV^2 \frac{v}{V} S(\omega^*) e^{i\omega(\frac{|\vec{x}|}{c_0} - t)}, \tag{58}$$

is the force per unit span b. Using this expression to calculate $\mathcal{P}$, we get

$$\mathcal{P} = (\rho_0 c_0^3 A) \left[\frac{\pi}{6} M^6 (\frac{v}{V})^2 (\frac{\omega c}{2V})^2 \frac{b^2}{A} \right] S\overline{S}. \tag{59}$$

5 The Quadrupole:

The concept of a dipole as a combination of two monopoles with equal but opposite strength can be generalized to define a quadrupole as a combination of two dipoles of equal but opposite strength separated by $\vec{\ell'} = \{\ell'_i\}$. Following the same procedure, we obtain

$$p' = \frac{1}{4\pi} \ell'_j \frac{\partial^2}{\partial x_i \partial x_j} (\frac{f_i(t - \frac{r}{c_0})}{r}). \tag{60}$$

If we introduce the tensor

$$T_{ij} = \ell'_j f_i. \tag{61}$$

The expression for the pressure (60) can be cast as

$$p' = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} (\frac{T_{ij}(t - \frac{r}{c_0})}{r}) \tag{62}$$

Note T_{ij} has the dimension of a shear $\times$ volume. The far field expression for a quadrupole is

$$p' = \frac{1}{4\pi r c_0^2} \ddot{T}_{ij} (t - \frac{r}{c_0}) \frac{x_i x_j}{r^2} \tag{63}$$

If $\vec{\ell'}$ is in the same direction as $\vec{f}$, the quadrupole is said to be longitudinal, and we have

$$p' = \frac{1}{4\pi r c_0^2} \frac{d^2 T}{dt^2} cos^2\theta, \tag{64}$$

where $T = |\vec{\ell'}| f$. On the other hand, if $\vec{\ell'}$ is normal to $\vec{f}$, the quadrupole is said to be lateral, and we have

$$p' = \frac{1}{4\pi r c_0^2} \frac{d^2 T}{dt^2} cos\theta sin\theta, \tag{65}$$

6 The Green's Function of the Wave Equation

For Laplace equation, the Green's function $G(\vec{x}, \vec{y})$ is solution to the inhomogeneous equation

$$\nabla^2 G = -4\pi\delta(\vec{x} - \vec{y}).. \tag{66}$$

Integrate 66 over a sphere centered at $\vec{y}$ of radius $r = |\vec{x} - \vec{y}|$,

$$\int_{\Sigma} \nabla^2 G \, d\vec{x} = -4\pi. \tag{67}$$

This can be rewritten using the divergence theorem as

$$\int_{\Sigma} \nabla^2 G \, d\vec{x} = \int_{\Sigma} \nabla G \cdot \vec{n} \, d\Sigma = \frac{dG}{dr} 4\pi r^2 = -4\pi$$

which gives the *free-space Green's function* for Laplace equation as

$$G = \frac{1}{r}. \tag{68}$$

The Green's function $G(\vec{x}, \vec{y}, t, \tau)$ for the wave equation is solution to

$$\left(\nabla^2 - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \right) G = -4\pi\delta(\vec{x} - \vec{y}) \, \delta(t - \tau). \tag{69}$$

Because of the spherical symmetry, the general solution to 69 is

$$G = \frac{f(t - \tau - \frac{r}{c_o})}{r}., \tag{70}$$

For the cases where $r \neq 0$, $t \neq \tau$, the function satisfies

$$\left(\nabla^2 - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \right) G = 0.$$

Near $r = 0$,

$$\left(\nabla^2 - \frac{1}{c_o^2} \frac{\partial^2}{\partial t^2} \right) G \sim f(t - \tau) \nabla^2 \left(\frac{1}{r} \right) = -4\pi\delta(\vec{x} - \vec{y}) \, \delta(t - \tau).$$

Hence, $f(t) = \delta(t)$ and we have the following expression for the Green's function

$$G = \frac{\delta \left(t - \tau - \frac{|\vec{x} - \vec{y}|}{c_o} \right)}{|\vec{x} - \vec{y}\,|}. \tag{71}$$

6.1 Harmonic Time-Dependence

For a harmonic time-dependence, we seek solutions of the form

$$\phi(\vec{x},t) = \varphi(\vec{x})e^{-i\omega t}. \tag{72}$$

The function φ satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\varphi = 0, \tag{73}$$

where $k = \omega/c_0$. The Green's function for 73 is solution to

$$(\nabla^2 + k^2)g(\vec{x},\vec{y}) = -4\pi\delta(\vec{x}-\vec{y}). \tag{74}$$

It is readily seen that

$$g(\vec{x},\vec{y}) = \frac{e^{\pm ikr}}{r}. \tag{75}$$

This can be also obtained by taking the Fourier transform in time of 71 and noting that

$$\frac{e^{-i(\omega t \pm kr)}}{r} = \frac{1}{r}\int_{-\infty}^{+\infty} \delta\left(t-\tau \pm \frac{r}{c_o}\right) e^{-i\omega\tau} d\tau. \tag{76}$$

7 Distribution of Sources and Forces

The linearized Euler equations for non-viscous non-heat-conducting fluid with no-mean motion are

$$\rho_o \frac{\partial}{\partial t}\vec{v} = \vec{f} - \nabla p' \tag{77}$$

$$\frac{\partial \rho'}{\partial t} + \nabla\cdot\rho_o\vec{v} = q \tag{78}$$

where $\vec{f}$ is a force per unit volume and q is a mass flow rate per unit volume.
Combining the two equations gives

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) p' = -\frac{\partial q}{\partial t} + \nabla\cdot\vec{f}. \tag{79}$$

In order to find solutions to the acoustic wave equation satisfying 79, we consider the inhomogeneous equation

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right) p' = -h(\vec{x},t). \tag{80}$$

Using the Green's function 71, and noting that

$$h(\vec{x},t) = \int h(\vec{y},\tau)\delta(\vec{x}-\vec{y})\delta(t-\tau)d\vec{y}d\tau, \tag{81}$$

we get

$$p' = \frac{1}{4\pi} \int \frac{h(\vec{y}, t - \frac{r}{c_0})}{r} d\vec{y} + p'_0, \tag{82}$$

where p'_0 is a solution to the homogeneous equation and accounts for the effects of boundaries. In what follows we focus on the inhomogeneous solution.

7.1 Source Distribution

In this case $h = \frac{\partial q}{\partial t}$. Hence, we have

$$p' = \frac{1}{4\pi} \int \frac{\frac{\partial q}{\partial t}(\vec{y}, t - \frac{r}{c_0})}{r} d\vec{y}, \tag{83}$$

7.1.1 Single Source

For a single source at $\vec{y_0}$, $q(\vec{y}, t) = m(t)\delta(\vec{y} - \vec{y_0})$. Substituting into 83 , we get back the pressure expression 30 for a single source.

$$p' = \frac{\dot{m}(t - \frac{r_0}{c_0})}{4\pi r_0}, \tag{84}$$

where $r_0 = |\vec{x} - \vec{y_0}|$.

7.2 Force Distribution

Force a distribution $\{f_i\}$, we consider the solution to the equation

$$\left(\nabla^2 - \frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}\right)\varphi_i = f_i. \tag{85}$$

This gives

$$\varphi_i = -\frac{1}{4\pi} \int \frac{f_i(\vec{y}, t - \frac{r}{c_0})}{r} d\vec{y}. \tag{86}$$

Taking the divergence of (86), we get

$$p' = -\frac{1}{4\pi}\frac{\partial}{\partial x_i} \int \frac{f_i(\vec{y}, t - \frac{r}{c_0})}{r} d\vec{y}. \tag{87}$$

A more direct expression for the pressure is given by

$$p' = \frac{1}{4\pi} \int [\frac{\frac{\partial f_i}{\partial t}(\vec{y}, t - \frac{r}{c_0})}{rc_0} + \frac{f_i(\vec{y}, t - \frac{r}{c_0})}{r^2}]\frac{(x_i - y_i)}{r} d\vec{y}. \tag{88}$$

The acoustic field is given by

$$p' = \frac{1}{4\pi c_0} \int [\frac{\frac{\partial f_i}{\partial t}(\vec{y}, t - \frac{r}{c_0})}{r}]\frac{(x_i - y_i)}{r} d\vec{y}. \tag{89}$$

7.2.1 Single Force

For a single force at $\vec{y_0}$, $f_i(\vec{y},t) = F_i(t)\delta(\vec{y}-\vec{y_0})$. Substituting into (88) , we get back the pressure expression (30) for a single source,

$$p' = \frac{1}{4\pi}[\frac{\frac{dF_i}{dt}(t-\frac{r_0}{c_0})}{r_0 c_0} + \frac{F_i(t-\frac{r_0}{c_0})}{r_0^2}]\frac{(x_i - y_{0i})}{r_0}. \tag{90}$$

The acoustic field is

$$p' = \frac{1}{4\pi r c_0}\frac{dF}{dt}(t-\frac{r_0}{c_0})cos\theta, \tag{91}$$

where F is the magnitude of the force $\mathbf{F}$ and θ is the angle between $\mathbf{F}$ and the observer direction $\mathbf{x}$. Note that as expected these results are identical to those obtained for a dipole. This confirms that a dipole represents the field of a force.