Acoustic Waves in a Duct

1 One-Dimensional Waves

The one-dimensional wave approximation is valid when the wavelength λ is much larger than the diameter of the duct D,

$$\lambda \gg D.$$

The acoustic pressure disturbance p' is then governed by

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial x^2} = 0, \tag{1}$$

where c is the speed of sound. In air at $293^o K$, $c = 343 m/s$. The general solution of (1) is of the form

$$p'(x,t) = f\left(t - \frac{x}{c}\right) + g\left(t + \frac{x}{c}\right) \tag{2}$$

For harmonic waves, the general solution is of the form

$$p'(x,t) = Ae^{-i\omega\left(t-\frac{x}{c}\right)} + Be^{-i\omega\left(t+\frac{x}{c}\right)}. \tag{3}$$

In order to determine a particular solution to (1), we need to specify the initial and boundary conditions associated with a real problem. For example, let us consider a piston located at $x = 0$ in a semi-infinite duct as shown in Figure 1.

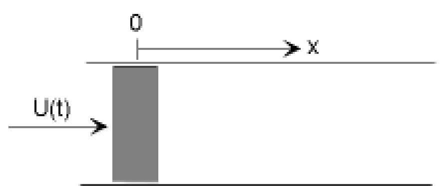

Figure 1: One-Dimensional Wave Propagation in a Duct

At time $t = 0$, the piston begins to oscillate about its mean position with a speed $U(t)$. Since the duct extends from 0 to ∞, the physical principle of *causality* tells us that all waves must propagate from the source of sound outward, i.e., from the piston to the right. This is equivalent to a homogeneous boundary condition imposed at infinity. As a result,

$$g\left(t + \frac{x}{c}\right) \equiv 0.$$

The fluid velocity u is related to the acoustic pressure p' by the momentum equation

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

This gives

$$u(x,t) = \frac{p'(x,t)}{\rho c} = \frac{1}{\rho c}\, f\left(t - \frac{x}{c}\right) \tag{4}$$

We now apply the boundary condition at $x = 0$,

$$u(t,0) = U(t) = \frac{1}{\rho_o c} f(t) \quad \text{for} \quad t \geq 0.$$

and solving for f gives

$$f\left(t - \frac{x}{c}\right) = \rho c\, U\left(t - \frac{x}{c}\right) \quad \text{for} \quad t \geq \frac{x}{c}.$$

Note that the initial condition $u(0,x) = 0$ for $x > 0$ implies that the sound produced by the oscillation of the piston will not reach locations $x > ct$. This is equivalent to taking $U(t) \equiv 0$ for $t < 0$.
Substituting the expression of f into Equation (4) gives

$$p' = \rho c\, U\left(t - \frac{x}{c}\right).$$

Note that if the piston has been oscillating for a very long time, we can take the initial time at $-\infty$ instead of 0. In this case $U(t)$ can be defined for all t. This is the case for harmonic oscillations where $U(t) = Ae^{-i\omega t}$. The solution takes the form

$$p' = A\rho c e^{-i\omega\left(t - \frac{x}{c}\right)},$$

where A is the wave amplitude. Such a wave is called a *plane* wave since its phase is constant in a plane perpendicular to the x-axis. The physical solution is the real part (or the imaginary part) of this solution,

$$\boxed{p' = A\rho c \cos\left[\omega\left(t - \frac{x}{c}\right)\right]} \tag{5}$$

Note that $v = p'/(\rho c)$. The acoustic intensity

$$I = p'v = A^2\rho c cos^2[\omega(t - \frac{x}{c}c)]. \tag{6}$$

The average intensity $\bar{I}$ and acoustic power $\mathcal{P}$ can then readily calculated

$$\bar{I} = \frac{1}{2}A^2\rho c \tag{7}$$
$$\mathcal{P} = \frac{1}{2}A^2\rho c S, \tag{8}$$

where S is the duct cross section.

The following parameters are usually used to describe harmonic waves.

$$\begin{aligned}
&\text{Circular Frequency:} && \omega = 2\pi f \quad \text{[Radians per second]} && (9)\\
&\text{Frequency:} && f = \frac{\omega}{2\pi} \quad \text{[Hertz=cycles per second]} && (10)\\
&\text{Period:} && T = \frac{1}{f} = \frac{2\pi}{\omega} \quad \text{[Seconds]} && (11)\\
&\text{Wavelength:} && \lambda = \frac{2\pi c}{\omega} && (12)\\
&\text{Wave Number:} && k = \frac{\omega}{c} = \frac{2\pi}{\lambda} && (13)\\
&\text{Phase Speed:} && c = \frac{\omega}{k}. && (14)\\
& && && (15)
\end{aligned}$$

2 Rectangular Duct with Rigid Boundaries

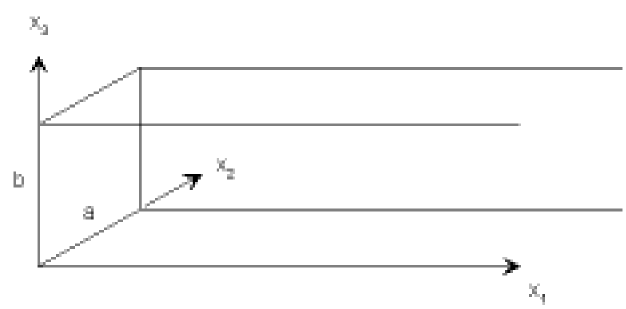

Figure 2: Wave Propagation in a Rectangular Duct

A schematic of the duct is shown in Figure 2. The governing equation for the acoustic pressure is similar to that of the one-dimensional case,

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0. \tag{16}$$

The rigid walls assumption gives the boundary condition $\frac{\partial p'}{\partial n} = 0$. We use the method of separation of variables to find a time-harmonic solution to this problem. Thus we assume a solution of the form

$$p'(\vec{x}, t) = f(x_1)\, g(x_2)\, h(x_3)\, T(t) \tag{17}$$

Substituting (17) into (16) gives

$$\frac{1}{c^2}\frac{T''}{T} - \frac{f''}{f} - \frac{g''}{g} - \frac{h''}{h} = 0. \tag{18}$$

If the oscillation frequency is ω, then

$$\frac{T''}{T} = -\omega^2 \qquad \Longrightarrow \qquad T(t) = e^{i\omega t}.$$

We could have considered a solution of the form $e^{-i\omega t}$. This would give the complex conjugate of the solution obtained with $e^{i\omega t}$. This, however,does not affect the physical solution which is the real part of the mathematical solution.

Substituting the expression for T into (18) gives

$$\frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} = -\frac{\omega^2}{c^2}$$

If we assign $\frac{g''}{g}$ and $\frac{h''}{h}$ to be constant, we obtain

$$\begin{aligned} \frac{g''}{g} &= -\alpha^2 && \Longrightarrow && g(x_2) = A_1\cos(\alpha x_2) + B_1\sin(\alpha x_2) \\ \frac{h''}{h} &= -\beta^2 && \Longrightarrow && h(x_3) = C_1\cos(\beta x_3) + D_1\sin(\beta x_3) \end{aligned}$$

Applying the boundary condition at $x_2 = \{0, a\}$ and $x_3 = \{0, b\}$ gives:

$$B_1 = 0, \qquad \alpha a = m\pi,$$

$$D_1 = 0, \qquad \beta b = n\pi,$$

and

$$g(x_2) = A_1\cos\left(\frac{m\pi}{a}x_2\right), \qquad h(x_3) = C_1\cos\left(\frac{n\pi}{b}x_3\right),$$

where m and n are integers. Finally, we have

$$\frac{f''}{f} = -\frac{\omega^2}{c^2} + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

giving,

$$f(x_1) = E_1 e^{-ik_{mn}x_1} + F_1 e^{ik_{mn}x_1},$$

where k_{mn} is defined as

$$k_{mn}^2 = \frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

Since we have assumed $\omega > o$, and we are considering acoustic waves propagating to the right, if we take

$$k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \tag{19}$$

then $E_1 = 0$. Equation (19) which gives the wave number k_{mn} in terms of the frequency ω is known as the *dispersion equation.* From the solutions for T, f, g, and h, and taking $F_1 = 1$, we have the solution

$$p'_{mn}(\vec{x}, t) = \cos\left(\frac{m\pi x_2}{a}\right)\cos\left(\frac{n\pi x_3}{b}\right)e^{i(k_{mn}x_1 - \omega t)}. \tag{20}$$

p'_{mn} is referred to as the (m, n) *mode*. The velocity in the x_1 direction is

$$v_{1mn} = \frac{p'_{mn}}{Z_{mn}}, \tag{21}$$

where the impedance

$$Z_{mn} = \frac{\rho c}{\sqrt{1 - (\frac{m\pi c}{a\omega})^2 - (\frac{n\pi c}{b\omega})^2}}. \tag{22}$$

The general solution is of the form

$$p'(\vec{x}, t) = \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} a_{mn} p'_{mn}(\vec{x}, t) \tag{23}$$

where a_{mn} are constants to be determined from the boundary conditions at $x_1 = 0$. If for example the acoustic waves are caused by the oscillations of a membrane described by $U_1(x_2, x_3) exp(-i\omega t)$, with the following Fourier ewxpansion

$$U_1(x_2, x_3) = \sum_{m=0}^{m=\infty} \sum_{n=0}^{n=\infty} b_{mn} \cos\left(\frac{m\pi x_2}{a}\right) \cos\left(\frac{n\pi x_3}{b}\right), \tag{24}$$

then

$$a_{mn} = b_{mn} Z_{mn}. \tag{25}$$

When

$$\omega < c[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]$$

k_{mn} is purely imaginary and the amplitude of the wave would either increase or decrease exponentially. Since growing waves are unacceptable physically, the waves will decay exponentially. These waves are called *evanescent* waves. As ω increases a new $\{m, n\}$ mode begins to propagate or is said to *cut on*. Thus at a given frequency ω, once the transients represented by the evanescent modes decay, the solution will have only a finite number of propagating modes.

Note that the dispersion equation is not linear, i.e., k_{mn} is not a linear function of ω. The axial phase speed, $c_p = \frac{\omega}{k_{mn}} > c$ and depends on ω. Waves with different frequencies have different phase speeds and as a result the waves disperse as they propagate. Such waves are called *dispersive waves*. The energy of dispersive waves propagate with the group velocity c_g defined as

$$c_g = \frac{d\omega}{dk} = c^2 \frac{k_{mn}}{\omega} = \frac{c^2}{c_p} < c.$$

If we assume for simplicity that $a = b$. Then

$$k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}(m^2 + n^2)} \tag{26}$$

Only waves where $\omega > \pi \frac{c}{a}(m^2 + n^2)^{\frac{1}{2}}$ will propagate. Introducing the wave length $\lambda = \frac{2\pi c}{\omega}$, gives the following condition for a wave to propagate,

$$a > \frac{\lambda}{2}\Big(m^2 + n^2\Big)^{\frac{1}{2}}.$$

If $a < \frac{\lambda}{2}$, only the (0,0) mode corresponding to a plane wave propagates in the duct. The higher order modes $(m > 0, n > 0)$ are *cut-off*. Thus if the duct diameter is much smaller than the wavelength the problem is reduced to that of the one-dimensional wave approximation. This result justifies the one-dimensional approximation for ducts with a diameter small compared to the wavelength.